

Exercises for Unit V (Introduction to non – Euclidean geometry)

V.1 : Facts from spherical geometry

Supplementary background readings.

Ryan : pp. 84 – 123

Exercises to work.

[**Note** : Hints for the first two exercises are given in math133f07update08.pdf .]

1. Two points on a circle Γ are said to be **diametrically opposite** if the center of the circle lies between them. Suppose that **A** and **B** are points on Γ that are not diametrically opposite, and let **M** and **m** be the major and minor arcs of Γ determined by **A** and **B**. Prove that **M** contains pairs of diametrically opposite points, but **m** does not contain any pairs of diametrically opposite points.
2. A basic theorem of geometry states that the intersection of a sphere and a plane is either empty, a single point or a circle. Find the radius of the circle of intersection for the sphere with equation $x^2 + y^2 + z^2 = 1$ and the plane with equation $x + y + z = 1$.
3. Suppose that we are given two spheres Σ_1 and Σ_2 which have different centers and at least one point in common. Prove that their intersection is contained in a plane Π which is perpendicular to the line joining their centers, and furthermore $\Sigma_1 \cap \Sigma_2 = \Sigma_1 \cap \Pi = \Sigma_2 \cap \Pi$. [**Hint**: Use vectors.]
4. Suppose that the sphere Σ and the plane **P** intersect in exactly one point **X**, and suppose that **Q** is the center of Σ . Prove that **QX** is perpendicular to **P**. Conversely, show that if the sphere Σ and the plane **P** intersect at **X** such that **QX** is perpendicular to **P**, then Σ and **P** have exactly one point in common. [**Hint**: These are analogs of the results for tangent lines to circles in plane geometry.]

V.2 : Attempts to prove Euclid's Fifth Postulate

Exercise(s) to work.

1. For each of the following theorems in plane geometry, give a proof that does **not** use Playfair's Postulate (**such a proof must be synthetic and not analytic or based upon linear algebra**).
 - (a) Let **A** and **B** be distinct points, and let x be a positive real number. Then there is a unique point **Y** on the open ray **(AB)** such that $d(\mathbf{A}, \mathbf{Y}) = x$.

Furthermore, we have $A*Y*B$ if and only if $x < d(A, B)$, and likewise we have $A*B*Y$ if and only if $x > d(A, B)$. [*Hint:* Use the Ruler Postulate.]

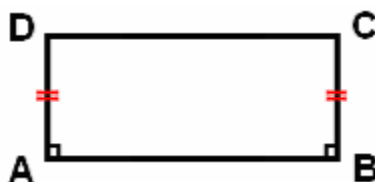
- (b) If L is a line and X is a point not on L , then there is a unique line through X which is perpendicular to L . [*Hint:* Let A and B be points on L , and find a point Y on the side of L opposite X such that $\triangle XAB \cong \triangle YAB$. Show that L is the perpendicular bisector of $[XY]$. Use the Exterior Angle Theorem to show there cannot be two perpendiculars from X with feet C and D .]
- (c) The Classical Triangle Inequality. [*Hint:* Given $\triangle ABC$, let D be such that $A*B*D$ and $d(B, D) = d(B, C)$. Show that $|\angle ADC| < |\angle ACD|$, and explain why the conclusion follows from this.]
- (d) The AAS Triangle Congruence Theorem. [*Hint:* Given $\triangle ABC$ and $\triangle DEF$ with $d(A, B) = d(D, E)$, $|\angle ABC| = |\angle DEF|$ and $|\angle ACB| = |\angle DFE|$, we know that $\triangle ABC \cong \triangle DEF$ if $d(A, C) = d(D, F)$, so assume this is false. Explain why, without loss of generality, we may assume $d(A, C) > d(D, F)$. Let G be a point of (AC) such that $d(A, G) = d(D, F)$, so that $\triangle ABG \cong \triangle DEF$. Why does this imply that $|\angle AGB| = |\angle ACB|$, and why does this contradict the Exterior Angle Theorem?]

V.3 : Neutral geometry

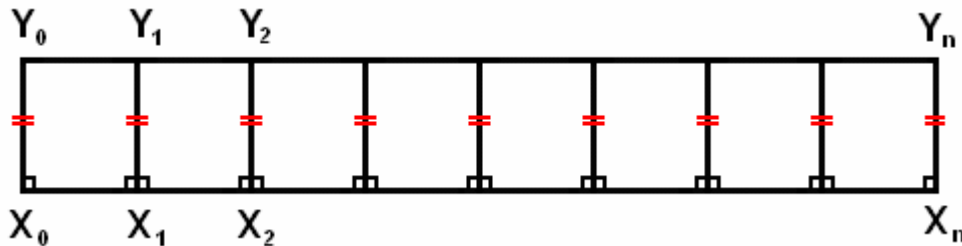
Exercises to work.

- Suppose that p and q are arbitrary positive real numbers. Prove that there is a Saccheri quadrilateral $\square ABCD$ with base AB such that $d(A, D) = d(B, C) = p$ and $d(A, B) = q$.
- Prove the following consequence of the Archimedean Law that was stated in the notes: *If h and k are positive real numbers, then there is a positive integer n such that $h/2^n < k$.* [*Hint:* Why is there a positive integer n such that $1/n < k/h$? Use this and the inequality $n < 2^n$.]

Standing hypotheses: In Exercises 2 – 5 below, points A, B, C, D in a neutral plane form the vertices of a Saccheri quadrilateral such that AB is perpendicular to AD and BC , and $d(A, D) = d(B, C)$. The segment $[AB]$ is called the **base**, the segment $[CD]$ is called the **summit**, and $[AD]$ and $[BC]$ are called the **lateral sides**. The vertex angles at C and D are called the **summit angles**.



3. Prove that the summit angles at **C** and **D** have equal measures. [**Hint:** Why do the diagonals have equal length? Use this fact to show that $\triangle BDC \cong \triangle ACD$.]
4. Prove that the line joining the midpoints of **[AB]** and **[CD]** is perpendicular to both **[AB]** and **[CD]**. [**Hint:** Imitate a similar argument from an earlier set of exercises for isosceles trapezoids in Euclidean geometry.]
5. Prove that $d(\mathbf{A}, \mathbf{B}) \leq d(\mathbf{C}, \mathbf{D})$. [**Hint:** First show by induction that if $\mathbf{G}_1, \dots, \mathbf{G}_n$ are arbitrary points, then $d(\mathbf{G}_1, \mathbf{G}_n) \leq d(\mathbf{G}_1, \mathbf{G}_2) + \dots + d(\mathbf{G}_{n-1}, \mathbf{G}_n)$. Next, show that one can find Saccheri quadrilaterals as in the picture below such that Saccheri quadrilateral **ABCD** is identical to Saccheri quadrilateral $\mathbf{X}_0\mathbf{X}_1\mathbf{Y}_1\mathbf{Y}_0$ in the picture, with all summits and all bases having equal lengths. The idea of the construction should be clear from the picture, but some work is needed to show that each $d(\mathbf{Y}_j, \mathbf{Y}_{j+1})$ is equal to $d(\mathbf{C}, \mathbf{D})$; it is helpful to start by considering the diagonals $[\mathbf{Y}_j \mathbf{X}_{j+1}]$ and the two triangles into which they split the Saccheri quadrilateral $\mathbf{X}_j\mathbf{X}_{j+1}\mathbf{Y}_{j+1}\mathbf{Y}_j$.



Using these observations, show that we have

$$nd(\mathbf{A}, \mathbf{B}) = d(\mathbf{X}_0, \mathbf{X}_n) \leq$$

$$d(\mathbf{X}_0, \mathbf{Y}_0) + d(\mathbf{Y}_0, \mathbf{Y}_1) + \dots + d(\mathbf{Y}_{n-1}, \mathbf{Y}_n) + d(\mathbf{Y}_n, \mathbf{X}_n) = nd(\mathbf{C}, \mathbf{D}) + 2d(\mathbf{A}, \mathbf{D}).$$

This means that $d(\mathbf{A}, \mathbf{B}) \leq d(\mathbf{C}, \mathbf{D}) + 2d(\mathbf{A}, \mathbf{D})/n$ for every positive integer n . Why can the second summand be made smaller than any positive real number, and why does this imply $d(\mathbf{A}, \mathbf{B}) \leq d(\mathbf{C}, \mathbf{D})$? Alternatively, why does this imply that $d(\mathbf{A}, \mathbf{B}) > d(\mathbf{C}, \mathbf{D})$ must be false?]

6. Suppose we are given Saccheri quadrilaterals $\square ABCD$ and $\square EFGH$ with right angles at **A, B** and **E, F** such that the lengths of the bases and lateral sides in $\square ABCD$ and $\square EFGH$ are equal. Prove that the lengths of the summits and the measures of the summit angles in $\square ABCD$ and $\square EFGH$ are equal.
7. Suppose are given Lambert quadrilaterals $\square ABCD$ and $\square EFGH$ with right angles at **A, B, C** and **E, F, G** such that $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{E}, \mathbf{F})$ or $d(\mathbf{B}, \mathbf{C}) = d(\mathbf{F}, \mathbf{G})$. Prove that $d(\mathbf{C}, \mathbf{D}) = d(\mathbf{G}, \mathbf{H})$, $d(\mathbf{A}, \mathbf{D}) = d(\mathbf{E}, \mathbf{H})$, and $|\angle CDA| = |\angle GHE|$.
8. In the setting of Exercise 5, prove that if we have $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{C}, \mathbf{D})$ then the Saccheri quadrilateral $\square ABCD$ is a rectangle.
9. Suppose that the points **A, B, C, D** in a neutral plane form the vertices of a Lambert quadrilateral with right angles at **A, B, C**. Prove that $d(\mathbf{A}, \mathbf{D}) \leq d(\mathbf{B}, \mathbf{C})$ and

$d(\mathbf{A}, \mathbf{B}) \leq d(\mathbf{C}, \mathbf{D})$. [*Hint:* Start by explaining why it suffices to prove the first of these. Show that there is a Saccheri quadrilateral $\square \mathbf{AEFD}$ such that \mathbf{B} and \mathbf{C} are the midpoints of the base $[\mathbf{AE}]$ and the summit $[\mathbf{FD}]$ respectively. Apply Exercise 4.]

10. In the setting of the preceding exercise, prove that if we have $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{C}, \mathbf{D})$ or $d(\mathbf{A}, \mathbf{D}) = d(\mathbf{B}, \mathbf{C})$, then the Lambert quadrilateral $\square \mathbf{ABCD}$ is a rectangle.

11. Suppose that \mathcal{P} is a neutral plane, and let p and q be arbitrary positive real numbers. Prove that there is a Lambert quadrilateral $\square \mathbf{ABCD}$ with right angles at \mathbf{A} , \mathbf{B} , and \mathbf{C} such that $d(\mathbf{A}, \mathbf{D}) = p$ and $d(\mathbf{A}, \mathbf{B}) = q$. [*Hint:* One can view a Lambert quadrilateral as half of a Saccheri quadrilateral.]

12. Suppose that \mathcal{P} is a neutral plane, and let p and q be arbitrary positive real numbers for which there is a Lambert quadrilateral $\square \mathbf{ABCD}$ with right angles at \mathbf{A} , \mathbf{B} , and \mathbf{C} such that $d(\mathbf{A}, \mathbf{B}) = p$ and $d(\mathbf{B}, \mathbf{C}) = q$. Let s be a positive real number such that $s < p$. Prove that there is a Lambert quadrilateral $\square \mathbf{WXYZ}$ with right angles at \mathbf{W} , \mathbf{X} , and \mathbf{Y} such that $d(\mathbf{W}, \mathbf{X}) = s$ and $d(\mathbf{X}, \mathbf{Y}) = q$. [*Hint:* Let \mathbf{E} be a point on (\mathbf{BA}) such that $d(\mathbf{B}, \mathbf{E}) = s$. Take the perpendicular line to \mathbf{AB} in \mathcal{P} containing \mathbf{E} , and explain why it must meet the diagonal segment (\mathbf{BD}) and the opposite side (\mathbf{CD}) using Pasch's Theorem.]

Note. For arbitrary positive real numbers p and q it is *not always possible* to find a Lambert quadrilateral $\square \mathbf{ABCD}$ with right angles at \mathbf{A} , \mathbf{B} , and \mathbf{C} such that $d(\mathbf{A}, \mathbf{B}) = p$ and $d(\mathbf{B}, \mathbf{C}) = q$. However, the proof involves a detailed analysis of the Poincaré model for hyperbolic geometry discussed in Sections V.5 – V.7 of the notes, and since our discussion of the model is only informal we cannot give a complete proof of this fact here. However, a drawing which depicts one such example (with $p = q$ sufficiently large) appears in the discussion of this exercise in the file

<http://math.ucr.edu/~res/math133/solutions7figures.pdf>

which can be found in the course directory.

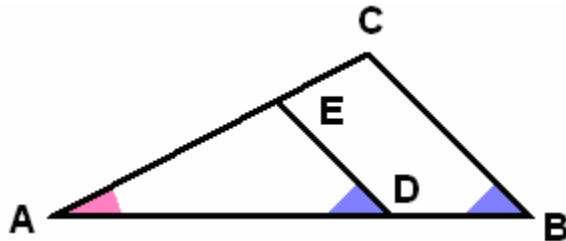
V.4 : Angle defects and related phenomena

Exercises to work.

1. Given a Saccheri quadrilateral in a hyperbolic plane \mathcal{P} , explain why the summit is always shorter than the base. If $\square \mathbf{ABCD}$ is a Lambert quadrilateral in a hyperbolic plane \mathcal{P} with right angles at \mathbf{A} , \mathbf{B} , and \mathbf{C} , what can one say about the lengths of the pairs of opposite sides $\{[\mathbf{AB}], [\mathbf{CD}]\}$ and $\{[\mathbf{BC}], [\mathbf{AD}]\}$? Give reasons for your answer.

2. Given a Saccheri quadrilateral in a hyperbolic plane \mathcal{P} , show that the line segment joining the midpoints of the summit and base is shorter than the lengths of the lateral sides.

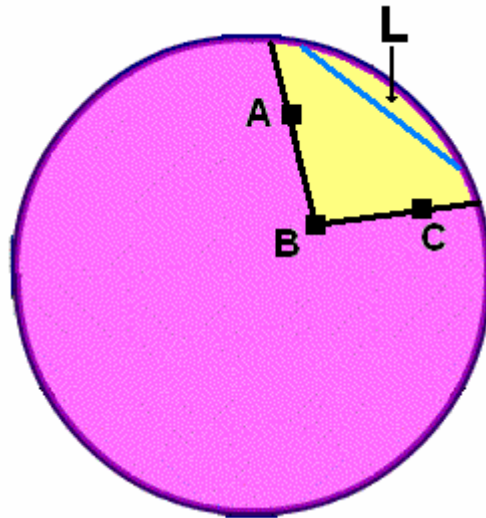
3. Suppose that ϵ is an arbitrary positive real number and \mathcal{P} is a hyperbolic plane. Prove that there is a triangle in \mathcal{P} whose angle defect is less than ϵ . [*Hint:* Let $\triangle ABC$ have defect δ . If one splits it into two triangles, why will **at least one** of them have defect at most $\frac{1}{2}\delta$? Show that if one iterates this enough times, one obtains the desired triangle.]
4. Suppose we are given an isosceles triangle $\triangle ABC$ in a hyperbolic plane \mathcal{P} with $d(A, B) = d(A, C)$, and let D and E be points on (AB) and (AC) such that $d(A, D) = d(A, E)$. Prove that $|\angle ABC| < |\angle ADE|$. [*Hint:* Compare the angle defects of the two isosceles triangles in the problem.]
5. Suppose we are given an equilateral triangle $\triangle ABC$ in a hyperbolic plane \mathcal{P} , and let D, E and F be the midpoints of $[BC]$, $[AC]$ and $[AB]$. Prove that $\triangle DEF$ is also an equilateral triangle and that $|\angle ABC| < |\angle DEF|$. Using the conclusion of the preceding exercise, explain why $\triangle AEF$ is **not** an equilateral triangle.
6. Prove the following result, which shows that Corollary III.2.15 (the “Third angles are equal” property) fails completely in hyperbolic geometry: Given a triangle $\triangle ABC$ in a hyperbolic plane \mathcal{P} and a point D on (AB) , then there exists a point E on (AC) such that $|\angle ABC| = |\angle ADE|$ but also $|\angle ACB| < |\angle AED|$.



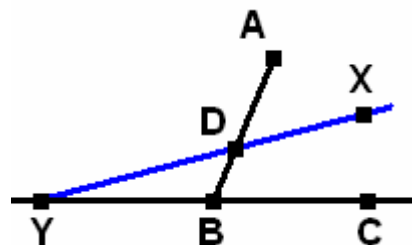
7. Suppose we are given a Saccheri quadrilateral $\square ABCD$ in a hyperbolic plane \mathcal{P} with base $[AB]$, and assume that the lengths of the base and lateral sides are equal. Does the ray $[AC$ bisect $\angle DAB$? Give reasons for your answer.
8. Suppose we are given a hyperbolic plane \mathcal{P} . Prove that there is a line L and an angle $\angle ABC$ in \mathcal{P} such that L is contained in the interior of $\angle ABC$. [*Hint:* Let B be a point not on L such that there are at least two parallel lines to L through B . If Y is the foot of the perpendicular from B to L and M is a line through B which is perpendicular to YB , then M is parallel to L , and there is also a second line N which is parallel to L . Explain why there is a ray $[BA$ on N such that A lies on the same side of M as Y , and explain why there is also a ray $[BC$ on M such that C and A lie on opposite sides of BY . Why is L disjoint from $\angle ABC$, why does the point Y on L lie in the interior of this angle, and why does this imply that the entire line is contained in the interior of $\angle ABC$?]

Note. In fact, it turns out that every angle $\angle ABC$ in a hyperbolic plane \mathcal{P} contains a line (in fact, infinitely many lines) in its interior. One way of seeing this is to use the Beltrami – Klein model for hyperbolic geometry described later in Section V.6. An illustration is given below; as noted in Section V.6, the points in the model are the points

inside the boundary circle, and the lines in the model are open chords whose endpoints lie on the boundary circle.



In this picture, $\angle ABC$ is given in the Beltrami – Klein model of the hyperbolic plane (the interior of the disk, with the boundary excluded), and L is just one example (among infinitely many) of a line which is contained entirely in the interior of $\angle ABC$. — This contrasts sharply with Euclidean geometry, where **a line containing a point of the interior of $\angle ABC$ must meet the angle in at one point.** [**Proof:** Suppose that L contains an interior point X of the angle; if A lies on L the result is true, so it suffices to consider the case where A does not lie on L . By Playfair’s Axiom we know that at least one of the lines BA and BC must have a point in common with L . Interchanging the roles of A and C if necessary, we may assume that BC meets L at some point. If this common point Y lies on $[BC$ we are done, so suppose it does not; in this case we know that B is between Y and C . The latter implies that Y and C lie on opposite sides of AB ; since X and C are assumed to lie on the same side of BA , it follows that the open segment (XY) meets AB at some point D . Since D is between X and Y and Y lies on BC , it follows that X and D lie on the same side of BC , and since A and X also lie on the same side of AC (by the assumption on X) it follows that the rays $[BA$ and $[BD$ are identical. But this means that L and $[BA$ have a point in common; thus we see that in all cases the line L has a point in common with $\angle ABC$. — The drawing below may be helpful for understanding the argument we have described in this paragraph.■]



V.5 : Further topics in hyperbolic geometry

Supplementary background readings.

Ryan : pp. 150 – 183

V.6 : Subsequent developments

Supplementary background readings.

Ryan : pp. 124 – 155

V.7 : Non – Euclidean geometry in modern mathematics

Supplementary background readings.

Ryan : pp. 150 – 183

Exercises to work.

1. Let \mathbf{a} be a vector in \mathbf{R}^2 , let $r > 0$, and let $\Gamma(r; \mathbf{a})$ denote the circle in \mathbf{R}^2 with center \mathbf{a} and radius r . The transformation

$$\mathbf{T}(\mathbf{v}) = \mathbf{a} + \frac{r^2}{|\mathbf{v} - \mathbf{a}|^2} \cdot (\mathbf{v} - \mathbf{a})$$

is called **inversion with respect to** $\Gamma(r; \mathbf{a})$. It is defined for all $\mathbf{v} \neq \mathbf{a}$, and its image is the same set. Prove that \mathbf{T} is $\mathbf{1} - \mathbf{1}$ onto, and in fact for all $\mathbf{v} \neq \mathbf{a}$ the point $\mathbf{T}(\mathbf{v})$ lies on the ray through \mathbf{v} with origin \mathbf{a} such that $|\mathbf{T}(\mathbf{v}) - \mathbf{a}| \cdot |\mathbf{v} - \mathbf{a}| = r^2$; using this description, explain why \mathbf{T} is equal to its own inverse.

2. Let \mathbf{T} be inversion with respect to the standard unit circle $\Gamma(\mathbf{1}; \mathbf{0})$ in the notation of the previous exercise, and let Γ_1 be the circle defined by the equation

$$(x - 1)^2 + (y - 1)^2 = 1.$$

Prove that the circles $\Gamma(\mathbf{1}; \mathbf{0})$ and Γ_1 meet in two points and that their tangent lines are perpendicular at these two points. Also, prove that if \mathbf{v} is a point of the circle Γ_1 then so is its image $\mathbf{T}(\mathbf{v})$. [**Hint:** It will probably be very helpful to graph the two circles before trying to solve this exercise.]