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Mathematics 133, Fall 2007, Examination 2

Answer Key

1. [*30 points*] (a) State Pasch's Posulate (also known as Pasch's Theorem).

(b) State the conditions that and ordered list of four coplanar points A, B, C, D must satisfy in order to be the vertices of a convex quadrilateral.

(c) Explain why the ordered list of points A, B, C, D will give the vertices of a convex quadrilateral if $AB||CD$ and $AD||BC$.

SOLUTION

(a) This is given in the notes.∎

(b) The points A and B must lie on the same side of CD, the points C and D must lie on the same side of AB, the points A and D must lie on the same side of BC, and the points B and C must lie on the same side of AD.∎

(c) If $AB||CD$, then A and B are on the same side of CD, and likewise C and D are on the same side of AB. Similarly, if $AD||BC$, then A and D are on the same side of BC, and likewise B and C are on the same side of AD.∎

2. [*25 points*] Suppose we are given two triangles $\Delta\, ABC$ and $\Delta\, DEF$ such that $\Delta\, ABC \cong \Delta\, DEF$, and let G and H be the midpoints of $[BC]$ and $[EF]$ respectively. Prove that $\Delta\, GAC \cong \Delta\, HDF$ and $\Delta\, GAB \cong \Delta\, HDE$.

SOLUTION

This is given in the solutions to the exercises for Section III.2.∎

3. [*25 points*] By definition and a theorem from the notes, the circumcenter V of $\Delta\, ABC$ is the unique point satisfying $|V-A|^2 = |V-B|^2 = |V-C|^2$, and the circumradius is the common value for the distance from V to these vertices. Find the circumcenter of the triangle in $\mathbf{R}^2$ with vertices $(0,0)$, $(3,4)$ and $(6,0)$.

10 POINTS EXTRA CREDIT. Find the circumradius of the triangle, and show it is a rational number by computing its value explicitly.

SOLUTION

In coordinates, the equations defining the circumradius are $x^2 + y^2 = (x-3)^2 + (y-4)^2 = (x-6)^2 + y^2$. If we subtract $x^2 + y^2$ from all sides of this equation we obtain the system of linear equations $0 = 25 - 6x - 8y = 36 - 12x$. Solving these, we obtain $x = 3$ and $y = 7/8$.

Extra credit question. The circumradius is equal to the distance from the circumcenter to the vertices. For computational purposes, in this problem it is most convenient to pick the vertex $(0,0)$, for in this case the circumradius simply turns out to be the length of V. Thus the circumradius is given by

$$\sqrt{3^2 + \left(\frac{7}{8}\right)^2} \quad = \quad \sqrt{\frac{24^2 + 7^2}{8^2}} \quad = \quad \sqrt{\frac{25^2}{8^2}}$$

and hence it is equal to $25/8 = 3\frac{1}{8}$.■

4. [*20 points*] Suppose that we are given an isosceles triangle $\Delta\, ABC$ such that $d(A,B) = d(A,C) > d(B,C)$. Determine which of the following is true, and state the theorem which yields this conclusion.

$$|\angle ABC| < 60^\circ < |\angle BAC| \ , \quad |\angle ABC| > 60^\circ > |\angle BAC|$$

SOLUTION

Since the longer side is opposite the larger angle, it follows that $|\angle ABC| > 60^\circ > |\angle BAC|$.■