FOUNDATIONS OF THE ANALYTIC APPROACH TO GEOMETRY

The ***points*** are the elements of the coordinate plane $\mathbb{R}^2$ or coordinate $3-$space $\mathbb{R}^3$.

The ***axioms*** are the axioms for set theory and for the real number system $\mathbb{R}$. The real number axioms can be classified as follows:

1. Standard properties of addition, subtraction, multiplication and division by nonzero numbers (commutative, associative and distributive laws, existence of 0 and 1 elements, existence of negatives and — for nonzero numbers — reciprocals).
2. Basic properties of ordering (positive numbers are closed under addition and multiplication, every number is positive, negative or zero, and furthermore these are mutually exclusive).
3. Completeness of the ordering (every bounded nondecreasing sequence converges to a limiting value).

All of ***the other basic data*** for Euclidean geometry are defined in terms of the real number system and the vector space structure on $\mathbb{R}^2$ or $\mathbb{R}^3$. Specifically, this is done as follows:

The **cosine function** is defined abstractly by means of the usual infinite series derived in calculus; many of its basic properties can be derived using this definition with no formal appeal to geometry (see pages 13 – 15 in the file **http://math.ucr.edu/~res/math133/verifications.pdf**). In particular, it follows that one can define an inverse cosine function on the interval $[0, \pi]$ with values in $[-1, 1]$.

Lines and ***planes*** are defined to be translates of $1-$ and $2-$ dimensional vector subspaces.

The ***distance*** between two points **p** and **q** is defined to be the length of the vector **p – q**. Given three collinear points **x**, **y** and **z**, we say that **y** is ***between*** **x** and **z** if the distances between the points satisfy $d(\mathbf{x}, \mathbf{z}) = d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$.

Given three noncollinear points **x**, **y** and **z**, the ***measurement*** θ ***of the angle*** $\angle \mathbf{xyz}$ is defined by the formula

$$\theta = \arccos\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)$$

where $\mathbf{a} = \mathbf{x} - \mathbf{y}$ and $\mathbf{b} = \mathbf{z} - \mathbf{y}$.

Plane areas and (in 3 dimensions) ***volumes*** are generally defined using the Lebesgue theory of integration (a refinement of Riemann integration, introduced in entry level graduate mathematics courses).