FOUNDATIONS OF THE
SYNTHETIC APPROACH TO GEOMETRY

BASIC DATA. The ***points*** in synthetic Euclidean geometry are the elements of some abstract set **P** (in the plane case) or **S** (in the $3-$dimensional case).

The other basic data types for synthetic Euclidean geometry are given by $(\mathcal{L}, *, d, \mu)$ in the plane case and $(\mathcal{P}, \mathcal{L}, *, d, \mu)$ in the $3-$dimensional case:

The ***lines*** in **P** (in the plane case) or **S** (in the $3-$dimensional case) are nonempty sets in a family $\mathcal{L}$ of nonempty subsets of **P,** and (in the $3-$dimensional case) the ***planes*** in **S** are a disjoint family $\mathcal{P}$ of nonempty subsets of **S.**

> This structure is enough to define the concepts of ***collinear and coplanar points*** (they are contained in some line or plane).

The ***betweenness relation*** on **P** or **S** is a three $-$ variable relationship **a**$*$**b**$*$**c** on the points of **P** or **S;** in other words, for each ordered triple of points **a, b, c** it is a statement about the points which is either true or false, and it is automatically false unless the three points are collinear and distinct (it mght or might not be true for a specific triple of distinct, collinear points). This relationship is often verbalized in the form, "The point **b** is between **a** and **c.**"

> The structure described up to this step is enough to define the concept of a ***convex set*** in **P** or **S.**

The ***distance function*** d on **P** or **S** is a function defined for every ordered pair of points in **P** or **S,** and it takes values in the set of all nonnegative real numbers.

The ***angle measurement function*** μ on **P** or **S** is a function defined for every ordered triple of noncollinear points in **P** or **S,** and it takes values in the set of all real numbers θ such that $0 < \theta < 180$. Its value is often denoted by $\mu(\angle \mathbf{xyz})$ or $|\angle \mathbf{xyz}|$.

> The structure described up to this step is enough to define nearly every remaining concept in classical Euclidean geometry which does not involve areas or volumes.

Areas and volumes. If we also want to study areas and volumes from the synthetic viewpoint, a few additional types of data are needed.

For each plane **Q** in **P** or **S** there is a family $\mathcal{M}_{\mathbf{Q}}$ of ***bounded measurable subsets*** in **Q,** and on each such family $\mathcal{M}_{\mathbf{Q}}$ there is an ***area function*** $\mathcal{a}$ which takes values in the nonnegative integers; in the $3-$dimensional case there is also a family $\mathcal{M}$ of ***bounded measurable subsets*** in **S,** and there is also a ***volume function*** $\mathcal{V}$ which is defined on $\mathcal{M}$ and takes values in the nonnegative integers.

<u>AXIOMS OR POSTULATES (*for our purposes these words have the same meaning*).</u> There are several groups of these assumptions**.**

The first three groups are specified in **http://math.ucr.edu/~res/math133/geometrynotes02a.pdf** (in each case**,** page number references are given)**:**

1. **Incidence Axioms:** $\mathbf{(I-1)}$ to $\mathbf{(I-6)}$**,** stated on page 37
2. **Betweenness Axioms:** $\mathbf{(B-1)}$ to $\mathbf{(B-2)}$**,** stated on page 45
3. **Plane Separation Axiom:** $\mathbf{(B-3)}$**,** stated on page 51

The next four groups are specified in **http://math.ucr.edu/~res/math133/geometrynotes02b.pdf** (in each case**,** page number references are given)**:**

4. **Distance Axioms:** $\mathbf{(D-1)}$ to $\mathbf{(D-3)}$**,** stated on page 59
5. **Angle Measurement Axioms:** $\mathbf{(AM-0)}$ to $\mathbf{(AM-3)}$**,** stated on page 64
6. **Triangle Congruence Axioms:** $\mathbf{(SAS)+(ASA)+(SSS)}$**,** stated on page 66
7. **Euclidean Parallelism Axiom (**Playfair's Postulate**):** $\mathbf{(P-0)}$**,** stated on page 80

The final two groups are specified in **http://math.ucr.edu/~res/math133/geometrynotes03c.pdf** (in each case**,** page number references are given)**:**

8. **Plane Area Axioms:** $\mathbf{(PM-1)}$ to $\mathbf{(PM-6)}$**,** stated on page 151
9. **Volume Axioms:** $\mathbf{(SM-1)}$ to $\mathbf{(SM-9)}$**,** stated on pages 161 – 162

Simplified sets of data and axioms

The formulation in this course is designed to develop the subject quickly and to avoid some issues that are elementary but tedious (and not particularly enlightening) to handle precisely**.** This is done by adding more data and assumptions than are logically necessary to develop classical Euclidean geometry synthetically**.** Generalities on axiom systems and streamlined lists of axioms for Euclidean geometry are discussed and presented in the following online file**:**

http://math.ucr.edu/~res/math153/history03c.pdf