

## V.7 : Non – Euclidean geometry in modern mathematics

Hyperbolic geometry, which was considered a dormant subject ... [in the middle of the 20<sup>th</sup> century], has turned out to have extraordinary applications to other branches of mathematics.

Greenberg, p. 382

It seems appropriate to conclude this unit on non – Euclidean geometry with a brief discussion of the role it plays in present day mathematics. Questions of this sort arise naturally, and In particular ***one might ask whether objects like the hyperbolic plane are basically formal curiosities or if they are important for reasons beyond just showing the logical independence of the Fifth Postulate.*** In fact, **hyperbolic geometry turns out to play significant roles in several contexts of independent interest. Some of these date back to the 19<sup>th</sup> century, but others were only discovered during the last few decades of the 20<sup>th</sup> century.**

### *Additional models for hyperbolic geometry*

Most of the ties between hyperbolic geometry and other topics in mathematics involve mathematical models for the hyperbolic plane (and spaces of higher dimensions) which are different from the Beltrami – Klein models described in the preceding section. There are three particularly important examples. One model (the ***Lorentzian model***) is discussed at length in Chapter 7 of Ryan, and two other basic models are named after H. Poincaré (pronounced *pwan – ca – RAY*). We shall only consider a few of properties of the Poincaré models in these notes. Further information can be found at the following online sites:

<http://www.geom.uiuc.edu/docs/forum/hype/model.html>

<http://www.mi.sanu.ac.yu/vismath/sazdanovic/hyperbolicgeometry/hypge.htm>

[http://math.fullerton.edu/mathews/c2003/poincairedisk/PoincareDiskBib/Links/PoincareDiskBib\\_Ink\\_1.html](http://math.fullerton.edu/mathews/c2003/poincairedisk/PoincareDiskBib/Links/PoincareDiskBib_Ink_1.html)

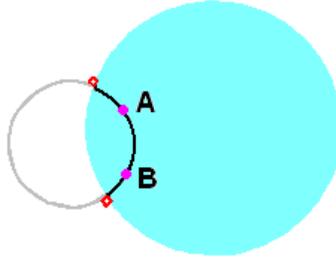
<http://mathworld.wolfram.com/PoincareHyperbolicDisk.html>

<http://www.geom.uiuc.edu/~crobles/hyperbolic/hypr/modl/>

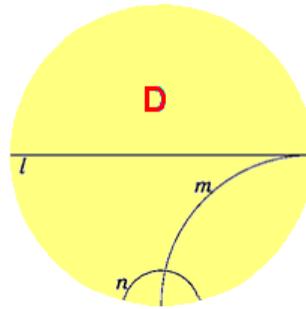
Probably the most important and widely used model for hyperbolic geometry is the ***Poincaré disk model***. In the 2 – dimensional case, one starts with the points which lie in the interior of a circle (*i.e.*, in an open disk) as in the Beltrami – Klein model, but the definitions of lines, distances and angle measures are different. The lines in this model are given by two types of subsets.

1. Open “diameter” segments with endpoints on the boundary circle.
2. Open circular arcs whose endpoints lie on the boundary circle and meet the boundary circle ***orthogonally*** (*i.e.*, at each endpoint, the tangent to the boundary circle is perpendicular to the tangent for the circle containing the arc).

An illustration of the second type of “line” is given below.



The drawing below illustrates several lines in the Poincaré disk model.



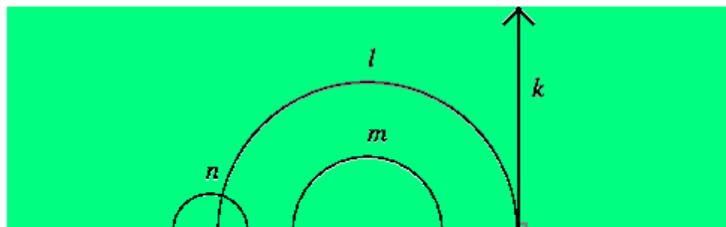
(Source: <http://www.geom.uiuc.edu/~crobles/hyperbolic/hypr/modl/pncr/>)

The Poincaré disk model distance between two points is given by a formula which resembles the comparable identity for the Beltrami – Klein model, and it is given in the first online reference in the list of online sites at the beginning of this section. On the other hand, one fundamentally important feature of the Poincaré disk model is that **its angle measurement is exactly the same as the Euclidean angle between the two intersecting curves** (*i. e.*, given by the usual angle between their tangents); such angle measure preserving models are said to be **conformal**. In contrast, both the distance and the angle measurement in the Beltrami – Klein model are different from their Euclidean counterparts.

The second Poincaré model in two dimensions is the **Poincaré half – plane model**, and its points are given by the points in the upper half plane of  $\mathbb{R}^2$ ; in other words, the points are all ordered pairs  $(x, y)$  such that  $y > 0$ . The lines in this model are once again given by two types of subsets.

- (1) Vertical open rays whose endpoints lie on the  $x$  – axis.
- (2) Open semicircular arcs whose endpoints lie on the  $x$  – axis.

The drawing below illustrates several lines in the Poincaré half – plane model.



(Source: <http://www.geom.uiuc.edu/~crobles/hyperbolic/hypr/modl/uhp/>)

The Poincaré half – plane model distance between two points is given by a formula in the first reference in the list of online sites at the beginning of this section. As in the preceding case, one fundamentally important feature of the Poincaré half – plane model is that its angle measurement is exactly the same as the Euclidean angle between two intersecting curves (*i.e.*, given by the usual angle between their tangents).

**Euclidean models of the hyperbolic plane.** In all the preceding models, it was necessary to introduce a special definition of distance in order to make everything work right. It would be very satisfying if we could give a nice model for the hyperbolic plane in Euclidean  $\mathbf{3}$  – space for which the distance is something more familiar (*i.e.*, the hyperbolic distance between two points is the length of the shortest curve in the model joining these points), but unfortunately this is not possible. The first result to show that no reasonably nice and simple model can exist was obtained by D. Hilbert (1862 – 1943) in 1901, and it was sharpened by N. V. Efimov (1910 – 1982) in the 1950s. One reference for Hilbert’s Theorem is Section **5 – 11** in the following book:

M. Do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice – Hall, Upper Saddle River, NJ, 1976. ISBN: 0–132–12589–7.

In contrast, during 1950s N. H. Kuiper (1920 – 1994) proved a general result which shows that the hyperbolic plane can be realized in Euclidean  $\mathbf{3}$  – space with the “right” distance, but the proof is more of a pure existence result than a method for finding an explicit example, and in any case the results of Hilbert and Efimov show that any such example could not be described very simply. Kuiper’s result elaborates upon some fundamental results of J. F. Nash (1928 – ); another an extremely important general result of Nash implies that the hyperbolic plane can be realized nicely in Euclidean  $n$  – space if  $n$  is sufficiently large; it is known that one can take  $n = \mathbf{6}$ , but apparently there are open questions about the existence of such realizations if  $n = \mathbf{5}$  or  $\mathbf{4}$ . Here are references for the realizability of the hyperbolic plane in Euclidean  $\mathbf{6}$  – space; the first is the original paper on the subject, and the second contains a fairly explicit construction of a nice model near the end of the file.

D. Blanuša, *Über die Einbettung hyperbolischer Räume in euklidische Räume*. Monatshefte für Mathematik **59** (1955), 217 – 229.

[http://www.math.niu.edu/~rusin/known-math/99/embed\\_hyper](http://www.math.niu.edu/~rusin/known-math/99/embed_hyper)

In yet another — and more elementary — direction, **it is not particularly difficult to represent SMALL PIECES of the hyperbolic plane nicely in Euclidean  $\mathbf{3}$  – space.** In particular, this can be done using a special surface of revolution known as a ***pseudosphere***. Further information on this surface can be found in many differential geometry books and notes, including pages 96 – 97 of the following online reference:

<http://math.ucr.edu/~res/math138A/dgnotes2006.pdf>

**Footnote.** (*This is basically nonmathematical information.*) The extraordinary life of John Nash received widespread public attention in the biography, ***A Beautiful Mind***, by S. Nasar, and the semifictional interpretation of her book in an Academy Award winning film of the same name. During the 1950s Nash proved several monumental results in geometry, but in nonmathematical circles he is better known for his earlier work on game theory, for which he shared the **1994** Nobel Prize in Economics with J. Harsányi (1920 –

2000) and R. Selten (1930 – ); an ironic aspect of this is noted in the footnote at the bottom of page 565 in Greenberg.

### *Hyperbolic geometry models and differential geometry*

We have already mentioned Riemann’s approach to the classical non – Euclidean geometries, which views the latter as special types of objects now called **Riemannian geometry**. Numerous properties of hyperbolic  $n$  – spaces play fundamental roles in many aspects of that subject, including some that have seen a great deal of progress over the past three decades. Three books covering many of these advances are discussed in a relatively recent book review by B. Kleiner [Bull. Amer. Math. Soc. (2) **39** (2002), 273 – 279.]

### *The Poincaré disk model and functions of one complex variable*

The investigation of the symmetries of a given mathematical structure has always yielded the most powerful results.

E. Artin (1898 – 1962)

There also is an important connection between the models described above and the subject of **complex variables**. In the latter subject, one considers complex valued functions that are defined in a region of the complex plane and defines the concept of differentiability in complete analogy with the real case; specifically, given a complex valued function  $f$  defined in such a region, the **complex derivative** of  $f$  at a point  $c$  in this region is the limit of

$$\frac{f(z) - f(c)}{z - c}$$

as  $z$  approaches  $c$ , provided that the limit exists. Functions which have complex derivatives at all points are said to be **complex analytic**. Many results and examples involving differentiable functions from ordinary calculus have analogs for complex analytic functions; eventually two subjects become quite distinct, but a discussion of such matters is beyond the scope of these notes. Our objective here is to state the following important relationship between the Poincaré disk model and analytic function theory.

**Theorem 1.** *Given the Poincaré disk model of the hyperbolic plane, let  $W$  be the underlying set of points viewed as a region in the plane. Then a  $1 - 1$  correspondence  $\phi$  from  $W$  to itself is a hyperbolic isometry if and only if **either** (1)  $\phi$  and its inverse are complex analytic **or else** (2) the **complex conjugates** of  $\phi$  and its inverse are complex analytic.*

The proof of this result involves methods and results from the theory of complex variables. A detailed treatment appears in Section **VII.2** of the following textbook:

S. Lang, **Complex Analysis** (4<sup>th</sup> Ed., corrected 3<sup>rd</sup> printing). Springer – Verlag, New York, 2003. ISBN: 0–387–98592–1.

Theorem 1 reflects the importance of hyperbolic geometry in a branch of complex analysis called the theory of *Riemann surfaces*. The latter play important roles in many mathematical subjects. For example, the proof of Fermat’s Last Theorem, mainly by A. Wiles (1953 – ), uses the hyperbolic geometrical structures that exist on most Riemann surfaces (**Note:** This fact led to some misunderstanding of Wiles’ work in some nontechnical books on the subject; these are summarized and accurately analyzed in the online article [http://en.wikipedia.org/wiki/Marilyn\\_vos\\_Savant](http://en.wikipedia.org/wiki/Marilyn_vos_Savant)).

There are also important connections between the theory of hyperbolic symmetries and a subject called *inverse geometry*; the latter studies a class of transformations called *inversions*, and some of these inversions correspond to reflections in the hyperbolic plane or hyperbolic 3 – space. Further information on this subject can be found in the previously cited book by Wallace and West (mainly Section 5.5), Chapters 24 and 25 of the previously cited book by Moïse, and also in the following online references:

[http://en.wikipedia.org/wiki/Inversive\\_geometry](http://en.wikipedia.org/wiki/Inversive_geometry)

<http://www.maths.gla.ac.uk/~wws/cabripages/inversive/inversive0.html>

<http://www.maths.gla.ac.uk/~wws/cabripages/klein/models.html>

Some basic facts about inverse geometry are also included in the exercises for this unit.

### *Regular tessellations*

For some minutes Alice stood without speaking, looking out in all directions over the country ... “I declare it’s marked out just like a large chessboard ... all over the world — if this is the world at all.”

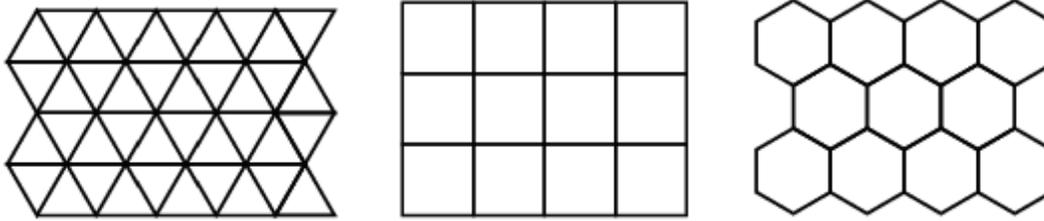
Lewis Carroll (C. L. Dodgson, 1832 – 1898),  
*Through the Looking Glass*

Although a precise and comprehensive description of hyperbolic geometry’s place in modern mathematics is beyond the scope of these notes, we shall describe one geometric manifestation of its role. However, before doing so we shall summarize the corresponding results for Euclidean and spherical geometry.

**The planar case.** A *regular tessellation* (or tiling) of the Euclidean plane is a decomposition of the plane into closed regions (see Section III.7) bounded by regular convex polygons such that the following hold:

1. All the bounding polygons have the same number of sides.
2. If the intersection of two distinct regions is nonempty, then it is a common side or vertex of the bounding polygons.

There are three obvious ways to construct such regular tilings of the Euclidean plane. If the regular polygons are squares, then one example corresponds to covering a flat surface by square tiles that do not overlap each other, and if the regular polygons are hexagons, then another example corresponds to the familiar honeycomb configuration of hexagons. A third example this type is the covering of a flat surface by tiles that are equilateral triangles. All of these are illustrated below.



Greek mathematicians (probably as early as the Pythagoreans) realized that the preceding examples were the only ones (up to similarity).

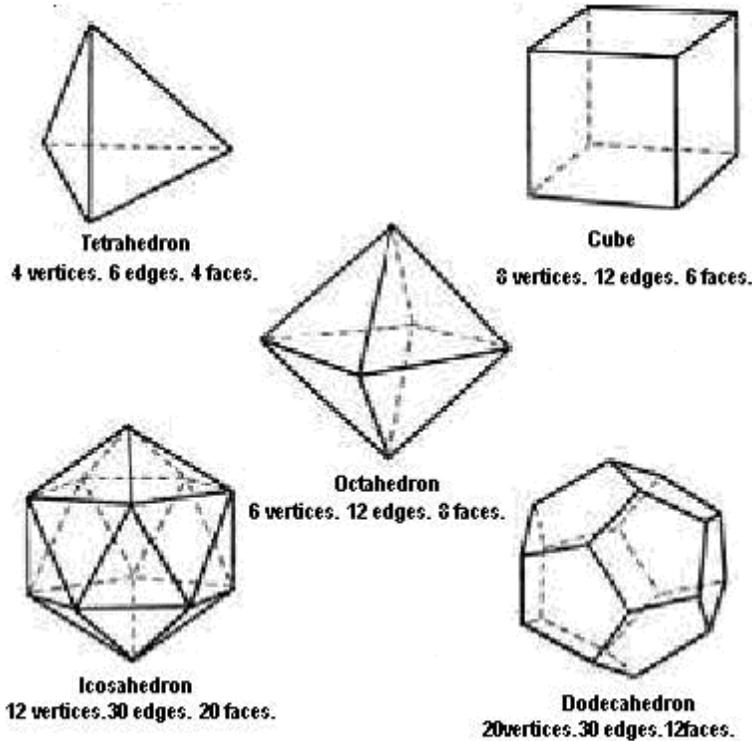
**The spherical case.** On the surface of the sphere, the regular tessellations correspond to **regular polyhedral** whose vertices lie on the sphere. More precisely, the vertices of the regular tessellation for the sphere are just the vertices of the regular polyhedron, its edges are the great circles joining two vertices (one for each pair of vertices which lie on a common edge of the polyhedron), and its faces are the regions bounded by the spherical polygons bounded by appropriate edges.

In view of the preceding discussion, the description of regular tessellations for the sphere reduces to the description of the possible types of regular polyhedra. The two simplest examples of regular polyhedra are a cube and a triangular pyramid such that each face is an equilateral triangle. Each of these illustrates the fundamental properties that all regular solids should have.

1. Every **2** – dimensional face should be a regular ***n*** – gon for some fixed value of  **$n \geq 3$** .
2. Every **1** – dimensional edge should lie on exactly two faces.
3. Every vertex should lie on ***r*** distinct faces for some fixed value of  **$r \geq 3$** .
4. No three vertices are collinear.
5. Given a face **F**, all vertices that are not on **F** lie on the same side of the plane containing **F**.

Regular polygons beyond triangles and squares were known in prehistoric times, and in fact archaeologists have also discovered early examples of stones carved and marked to represent several (in fact, most and maybe all) **3** – dimensional regular polygons. One major achievement of Greek mathematics (which appears at the end of Euclid’s **Elements**) was the proof that **there are exactly five distinct types of regular polyhedra**, and they are listed and illustrated below:

Type of polyhedron	No. of vertices	No. of edges	No. of faces
tetrahedron	4	6	2
cube	8	12	6
octahedron	6	12	8
icosahedron	12	30	20
dodecahedron	20	30	12



(Source: <http://www.goldenmeangauge.co.uk/platonic.htm>)

### Examples in the hyperbolic plane

The situation in the hyperbolic plane is entirely different. One important reason is given by the following result, which is also mentioned on page 176 of Ryan.

**Theorem 2.** *Let  $n$  be an integer greater than 2, and let  $\theta$  be a positive number less than  $180(n - 2)/n$ . Then there is a regular hyperbolic  $n$  – gon such that all the sides have equal length and the measures of all the vertex angles are equal to  $\theta$ .*

In particular, if  $n$  is greater than 4 and  $\theta = 90$ , then one might expect that we can form a regular tessellation of the hyperbolic plane with regular  $n$  – gons such that four meet at each vertex. In fact, this is possible. This is a special case of the following general result:

**Theorem 3.** *Suppose that  $m, n \geq 3$  are integers such that*

$$\frac{1}{m} + \frac{1}{n} < \frac{1}{2}.$$

*Then there is a regular tessellation of the hyperbolic plane into solid regular  $n$  – gons with  $m$  distinct polygons meeting at each vertex. Conversely, if there is a regular tessellation of the hyperbolic plane into solid regular  $n$  – gons with  $m$  distinct polygons meeting at each vertex, then the displayed inequality holds.*

There are several ways to prove this theorem. In particular, algebraic results of W. F. von Dyck (1856 – 1934) give an approach which is related to the viewpoint of Ryan's book.

Since there are infinitely many pairs of positive integers  $m$  and  $n$  satisfying these conditions, it follows that **there are infinitely many distinct regular tessellations of the hyperbolic plane**. The type of such a tessellation is generally denoted by the ordered pair  $(n, m)$ ; note that **the first coordinate gives the number of sides**. We shall give a few illustrations below; they are taken from the following sources:

<http://aleph0.clarku.edu/~djoyce/poincare/poincare.html>

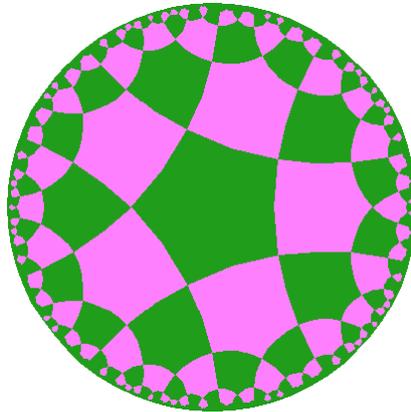
<http://www.hadron.org/~hatch/HyperbolicTesselations/>

<http://www.google.com/search?hl=en&q=hyperbolic+tesselations&btnG=Google+Search>

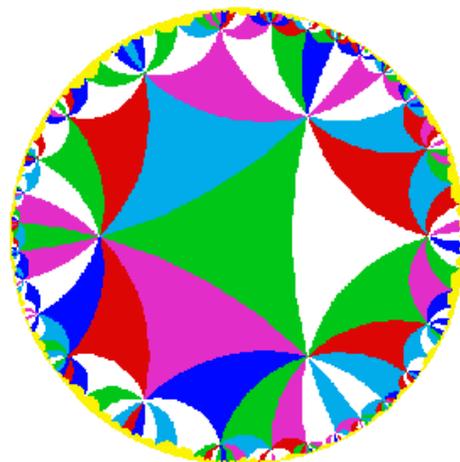
<http://www.btinternet.com/~connectionsinspace/Patterns and Space Filling g/Hyperbolic Geometry/body hyperbolic geometry.html>

<http://www.d.umn.edu/~ddunham/dunham04.pdf#search=%22hyperbolic%20tesselations%22>

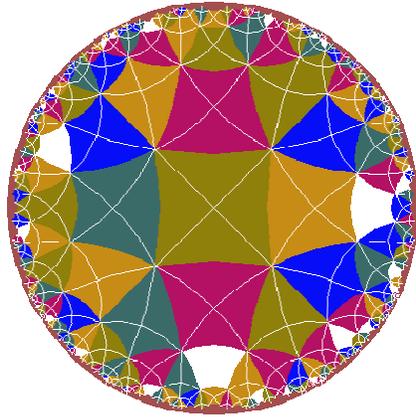
[http://en.wikipedia.org/wiki/Tilings\\_of\\_regular\\_polygons](http://en.wikipedia.org/wiki/Tilings_of_regular_polygons)



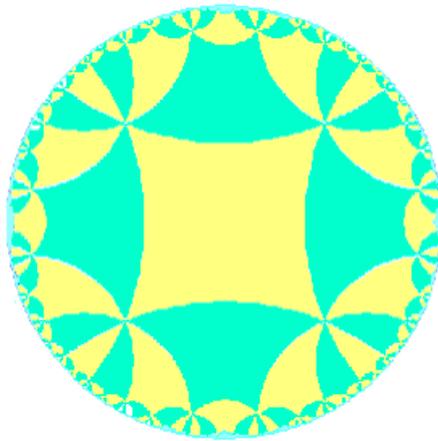
(5, 4)



(3, 12)

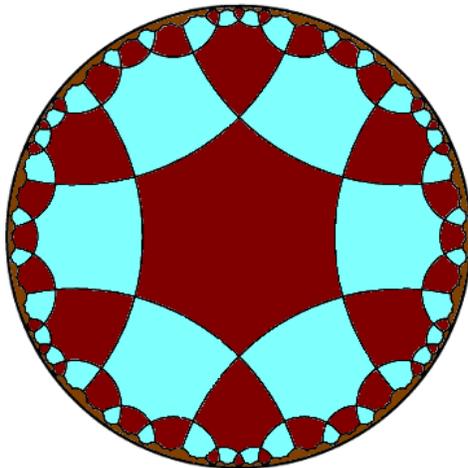


(4, 8)

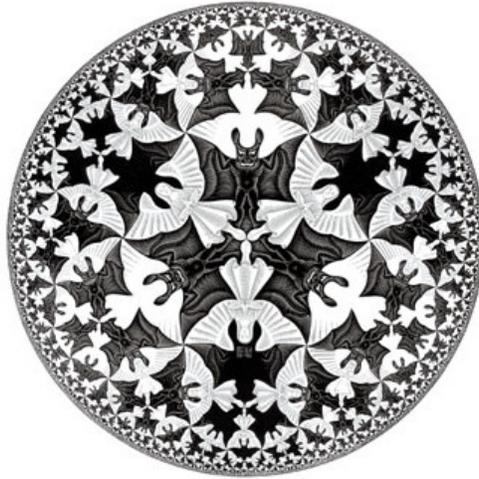


(4, 6)

Regular tessellations of the hyperbolic plane also appear in some of the artwork created by M. Escher (1898 – 1972). For example, the angels and devils in the picture **Circle Limit IV** fit together to form a tessellation by regular hexagons with right angles at every vertex (type (6, 4) in our notation). This can be seen from the illustrations below:



(Source: <http://www.d.umn.edu/~ddunham/mam/essay1.html>)



(Source: [http://www.allposters.com/-sp/Circle-Limit-IV-Posters\\_i96944\\_.htm](http://www.allposters.com/-sp/Circle-Limit-IV-Posters_i96944_.htm))

The following book contains further information on the interaction between art and progress in geometry during the past two centuries:

L. D. Henderson, *The Fourth Dimension and Non – Euclidean Geometry in Modern Art*. Princeton Univ. Press, Princeton, 1983. ISBN: 0–691–04008–7.

A recent summary of this book (which is currently out of print) is available at the following online site:

[http://www.space.com/scienceastronomy/generalscience/astrobizarre\\_fourD\\_010619.html](http://www.space.com/scienceastronomy/generalscience/astrobizarre_fourD_010619.html)

#### *Final remarks*

There is an interesting relationship between Theorem 3 and the results for regular tessellations of the sphere and the Euclidean plane. In the Euclidean plane, there is a regular tessellation into solid regular  $n$  – gons with  $m$  distinct polygons meeting at each vertex if and only if

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{2}.$$

because this equation holds if and only if  $(n, m)$  is equal to one of the three ordered pairs  $(3, 6)$ ,  $(4, 4)$  or  $(6, 3)$ . Similarly, on the sphere there is a regular tessellation of the hyperbolic plane into solid regular spherical  $n$  – gons with  $m$  distinct polygons meeting at each vertex if and only if

$$\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$$

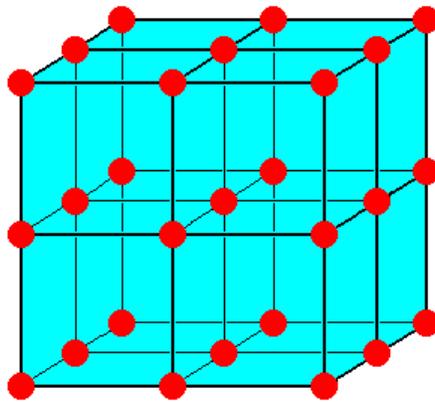
because this equation holds if and only if  $(n, m)$  is equal to one of the five ordered pairs  $(3, 3)$ ,  $(3, 4)$ ,  $(3, 3)$ ,  $(3, 5)$  or  $(5, 3)$ . If we combine these observations with Theorem 3, we obtain the following unified conclusion:

For each ordered pair  $(n, m)$  such that  $m, n \geq 3$ , there is a regular tessellation of either the **Euclidean plane**, the **hyperbolic plane** or the **sphere** into solid regular  $n$  – gons with  $m$  distinct polygons meeting at each vertex, and the specific type of plane supporting such a configuration is given by comparing  $\frac{1}{2}$  to the previously described sum of reciprocals:

$$\frac{1}{m} + \frac{1}{n}$$

In particular, the relevant geometry will be **spherical** if this sum is **greater than**  $\frac{1}{2}$ , it will be **Euclidean** if this sum is **equal to**  $\frac{1}{2}$ , and it will be **hyperbolic** if this sum is **less than**  $\frac{1}{2}$ .

**Honeycomb patterns in higher dimensions.** Clearly one can ask similar questions about regular decompositions of Euclidean **3** – space into nonoverlapping regular polyhedra which are congruent to each other, and the standard decomposition into cubes is a natural example which was probably known in prehistoric times (in the drawing below, the vertices for a portion of this decomposition are shown in red).



(Source: [http://www.metafysica.nl/turing/preparation\\_3dim\\_2.html](http://www.metafysica.nl/turing/preparation_3dim_2.html))

Tessellation – like decompositions of higher – dimensional spaces are often called **honeycombs** in the mathematical literature. It turns out that the obvious examples with cubes are the only types of regular honeycomb structures that can exist in Euclidean **3** – space, and the file

<http://math.ucr.edu/~res/math153/history08b.pdf>

discusses the impossibility of finding a decomposition into regular tetrahedra. Further information on this topic appears in Section **4.6** of the following classic text:

H. S. M. Coxeter, **Regular polytopes** (3<sup>rd</sup> Ed.). Dover Publications, New York, 1973.

Of course, one can also state similar questions for Euclidean spaces in dimensions greater than **3**, and the following online references give the answers to such questions:

[http://en.wikipedia.org/wiki/List\\_of\\_regular\\_polytopes#Tessellations\\_of\\_Euclidean\\_3-space](http://en.wikipedia.org/wiki/List_of_regular_polytopes#Tessellations_of_Euclidean_3-space)

[http://en.wikipedia.org/wiki/List\\_of\\_regular\\_polytopes#Tessellations\\_of\\_Euclidean\\_space](http://en.wikipedia.org/wiki/List_of_regular_polytopes#Tessellations_of_Euclidean_space)

Likewise, one can consider questions about such decompositions for hyperbolic spaces in dimensions greater than **2**. The online references cited above contain further information on this topic, and another basic source is Chapter **10** in the following book (see also Summary Table **IV** on page 213):

H. S. M. Coxeter, *The Beauty of Geometry: Twelve Essays*. Dover Publications, New York, 1999.

## V.8 : Summarizing the impact of non – Euclidean geometry

The last few sections of Unit **V** discuss the impact of non – Euclidean geometry on mathematics and related subjects from several different viewpoints. The following list summarizes the discussion:

1. In contrast to the laws of physics and chemistry, the laws for whole number arithmetic seem to be logically inevitable. However, the development of non – Euclidean geometry during the late 18<sup>th</sup> and most of the 19<sup>th</sup> centuries showed very conclusively that the classical setting for (Euclidean) geometry was not equally logically inevitable.
2. In many respects, the classical non – Euclidean geometry — which is often called *hyperbolic geometry* — behaves like a spherical geometry in which the radius of the sphere is an imaginary number.
3. Non – Euclidean was one of several important factors leading to the creation of a more rigorous logical foundation for mathematics during the late 19<sup>th</sup> and early 20<sup>th</sup> centuries. Mathematical models for both Euclidean and non – Euclidean geometry can be constructed within these frameworks for mathematics.
4. The development of non – Euclidean geometry eventually led to many alternative mathematical theories for describing the physical universe, including the currently standard models of Relativity Theory. There are some relationships between the latter and non – Euclidean geometry, but neither can really be described as part of the other.
5. For many (maybe most) practical purposes, Euclidean geometry is an extremely good first order approximation for studying small regions in any reasonable theory of space.
6. During the past half century, several mathematical advances have shown that hyperbolic geometry is fundamentally important. The subject has turned out to be more than just a logical curiosity, and many of its basic features continue to play important roles in several branches of mathematics and its applications. Two simply stated examples involve the proof of Fermat’s Last Theorem and several of Maurits C. Escher’s artistic creations.

In conclusion, it seems appropriate to end these notes with the following quotation from page **105** of Greenberg:

Let us not forget that no serious work toward constructing new axioms for Euclidean geometry had been done until the discovery of non – Euclidean geometry shocked mathematicians into reexamining the foundations of the former **Comment:** We have already noted that other considerations also played important roles in forcing a review of the foundations for classical geometry]. We have the paradox of non – Euclidean geometry helping us to better understand Euclidean geometry!