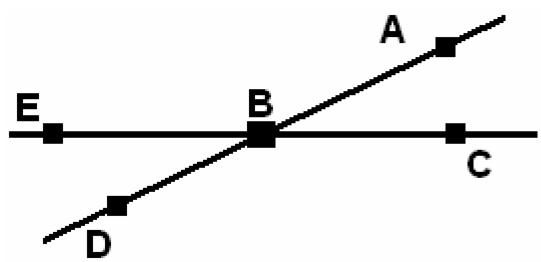
Irreducibility of ordered planes

The document http://math.ucr.edu/~res/math133/nonmetric_models.pdf states the theorem stated below**;** our purpose here is to give the proof together with some drawings which may be helpful for understanding the argument**.**

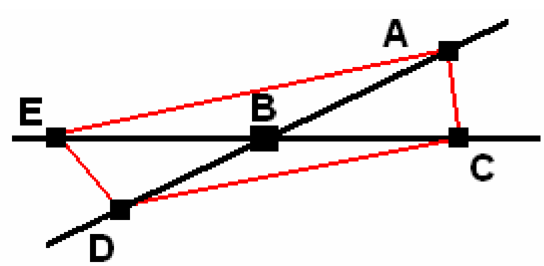
Theorem. *Suppose that* **P** *is plane which satisfies the standard axioms of incidence and order* (***i.e.***, *the betweenness and plane separation axioms*)*.* *If* **Q** *is a flat*, *noncollinear subset of* **P**, *then* **Q = P.**

For our purposes here, one especially important feature of such geometrical systems is that the standard **Crossbar Theorem** is true in all such geometrical systems (see Theorem **II.3.5** in the course notes)**;** proofs of this result using only the betweenness and separation axioms are is are given in pages 82 of Moïse and 116 – 117 of Greenberg**.**

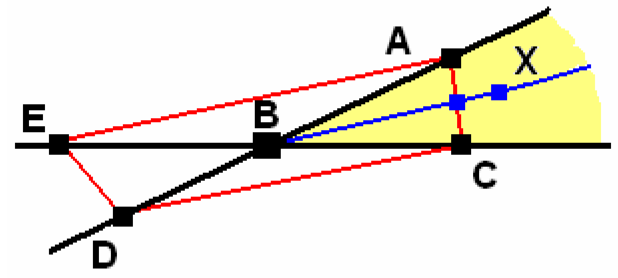
Proof. As in the related document http://math.ucr.edu/~res/math133/irreducibleplanes1.pdf (which proves the analogous result for all projective planes and all but one affine plane)**,** we are given that three noncollinear points **A, B, C** lie in the flat subset **Q.** It follows immediately that the lines **AB** and **BC** are both contained in **Q.**



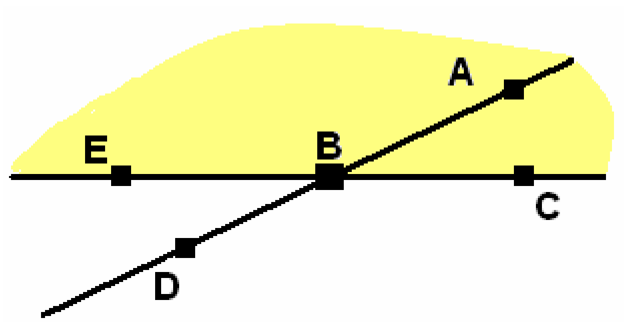
Starting with this**,** we need to show that every point of **P** must also lie in **Q.** There are several steps in this process**;** at each step we show that **Q** contains more points of **P** than were known at the preceding one**,** and ultimately we find that all points of **P** must be in **Q.** If we choose **D** and **E** such that **A**∗**B**∗**D** and **C**∗**B**∗**E** hold**,** then by flatness we know that both **D** and **E** must lie in **Q.**



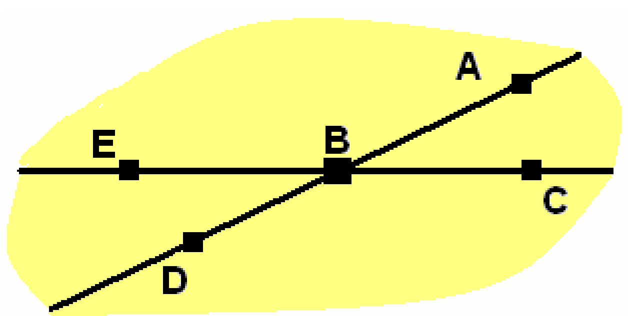
By flatness**,** it follows that the lines **AC, CD, DE** and **EA** are also contained in **Q,** and of course this means that the segments **(AC), (CD), (DE)** and **(EA)** are also contained in **Q.**



The next (very crucial**!**) step is to show that the entire interior of ∠**ABC** is contained in **Q,** and the drawing above suggests the argument**.** Given a point **X** in the interior of ∠**ABC,** the Crossbar Theorem implies that the ray **(BX** and the segment **(AC)** meet at some point **Y.** Since this point lies on **AC,** it must belong to **Q.** But we already know that **B** belongs to **Q,** and therefore the entire line **BY,** which contains **X,** must be contained in **Q.** Since **X** was an arbitrary point in the interior of ∠**ABC,** this proves the latter is contained in **Q.**



If we switch the roles of **C** and **E** in the preceding argument**,** we also find that the entire interior of ∠**ABE** is contained in **Q.** Combining this with previously derived information**,** we see that all points on the same side of **BC** as **A** must belong to **Q** (observe that such a point is either on **AB,** on the same side of **AB** as **C,** or on the same side of **AB** as **E**)**.**



Finally**,** if we switch the roles of **A** and **D** in the preceding argument**,** we also find that all points on the same side of **BC** as **D** must also belong to **Q.** Since all points of **P** either lie on **BC,** the same side of **BC** as **A,** or the same side of **BC** as **D,** it follows that every point in **P** must belong to **Q,** so that **Q** = **P.**■