Locus Problems

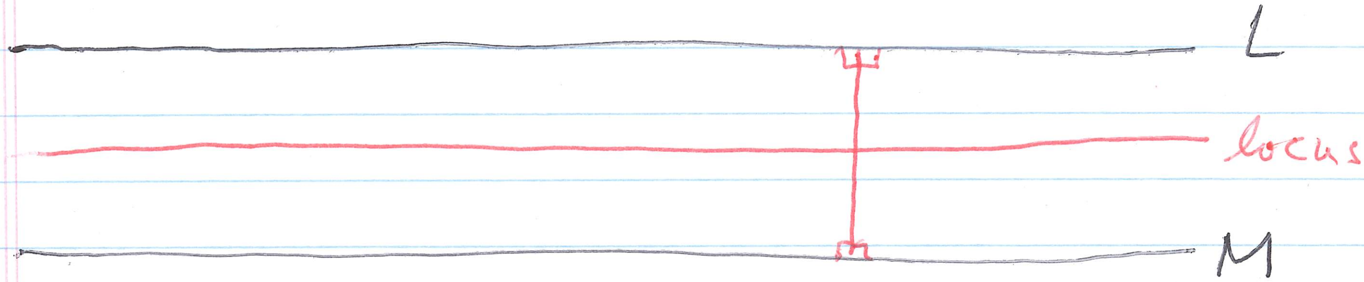
Classical Greek geometry contains a large number of problems of the following type.

Find the locus [= set] of points satisfying a list of conditions.

For example, one standard example is that the locus [set] of points that are equidistant from ~~both~~ two fixed points A and B (in some plane) is the perpendicular bisector line for the segment [AB].

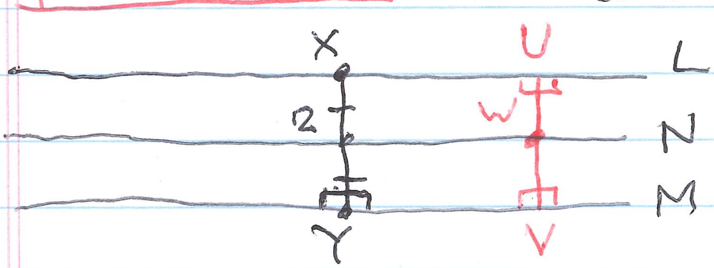
In their work on coordinate geometry, both Descartes and Fermat showed that some challenging problems of this sort could be solved fairly easily (at least relatively so!) if one uses coordinates. We shall illustrate this with a simple example, comparing the classical synthetic approach with the analytic approach.

<u>THEOREM</u>. The locus [set] of all points that are equidistant from two parallel lines (in the plane these two lines) is a third line parallel to the first two.



<u>Synthetic proof</u> Given $X \in L$ ~~and~~, drop a perpendicular from X to M, let Y be the foot of this perpendicular, and let Z be the midpoint of the segment $[XY]$. Let N be the unique line through Z which is parallel to both L and M. CLAIM N is the locus/set described in the statement of the theorem.

FIRST PART N is contained in the locus.



By construction, Z lies in the locus, so let $W \in N$, $W \neq Z$.

① The line XY is $\perp M$ and $M \parallel L$, so $XY \perp L$. Likewise, $XY \perp N$.

(2) If we drop a perpendicular from W to M, let V be its foot. As in (1), we have $WV \perp N$ and $WV \perp L$. Let U be the point where WV meets L.

(3) Since $W \in N$ and $W \neq Z$ = meeting point of XY and N, we know $W \notin XY$. Now $XY, WV \perp N \Rightarrow XY \parallel WV$.

(4) The preceding implies that we have rectangles $YVWZ$ and $XUWZ$. Since the opposite sides of rectangles have equal length, this means that the lengths satisfy

$$|WV| = |ZY| = |ZX| = |WU|.$$

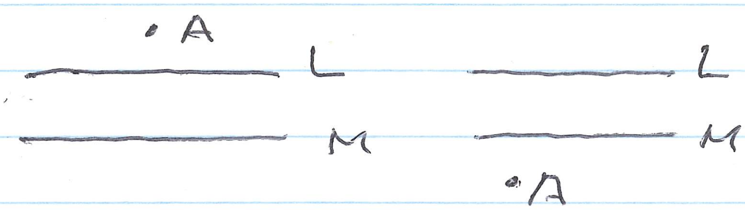
by construction!

Therefore W lies on the locus of points equidistant from L and M.

SECOND PART If A lies on the locus, then $A \in N$.

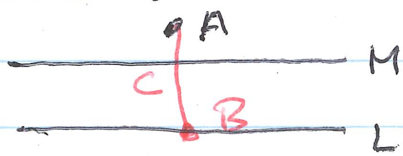
Step 1 A and L lie on the same side of M, and similarly A and M lie on the same side of L.

Claim neither of these can occur if A lies on the locus.



If we can prove the first statement, the second will follow by interchanging the roles of L and M in the argument, so it will suffice to prove A and L lie on the same side of M.

Suppose they don't.

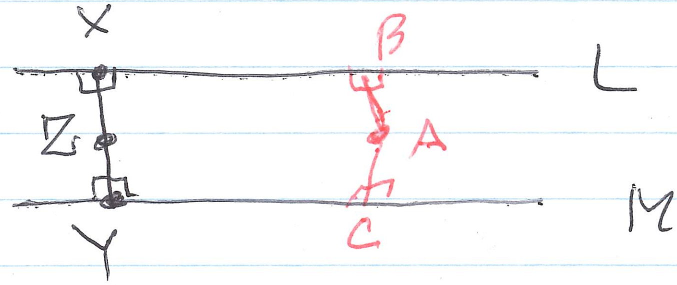


$A \notin L, M$ because of the locus cond. (e.g., $A \in L \Rightarrow$ $dist(A, L) = 0$ $dist(A, M) > 0$)

Drop a perpendicular from A to L with foot B. Then

$AB \perp L$ and $L \parallel M \Rightarrow AB \perp M$. Let C be the point where they meet. Since A and B are on opposite sides of M (because A & L are), C must be between A and B, so that $|AC| < |AB|$. This contradicts $dist(A, L) = dist(A, M)$, so the assumption that A & L lie on opposite sides must be false, and this finishes Step 1.

Step 2



Assume $A \neq Z$
(Know $Z \in$ locus & on N)

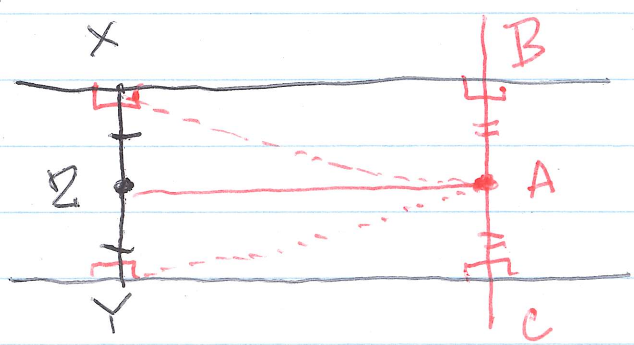
If we drop perpendiculars from A to L & M with feet B and C, then $AB = AC$.

① $AB \perp L$ and $XY \perp L \Rightarrow AB \parallel XY$

Likewise $AC \perp M$ and $XY \perp M \Rightarrow AC \parallel XY$.

② By the Euclidean Parallel Postulate, there is only one parallel to XY through A, so $AB = AC$.

Step 3



We have this

and we need to show $ZA = N$.

(1) We have a rectangle $YCBX$, so $|XB| = |YC|$ and $|XY| = |BC|$.

(2) By SAS, rt. Δ ~~BAX~~ $XBA \cong$ rt. ΔYCA. so that $|AX| = |AY|$.

(3) By the previously cited locus theorem, AZ is the perpendicular bisector for $[XY]$.

(4) Since $AZ \perp XY$ and $L \perp XY$, we have $AZ \parallel L$.

(5) By the Euclidean Parallel Postulate and $Z \in N \parallel L$, we have $AZ = N$, so that $A \in N$ as desired.

Thus if $A (\neq Z)$ is on the locus, then $A \in N$. ∎

<u>Analytic Proof</u> Choose coordinates so that M becomes the x-axis and hence L has the equation $y = c$ for some $c \neq 0$ (horizontal lines = lines $\parallel$ x-axis).

We can even position the y-axis so that $c > 0$.

Now let $P = \cancel{(x, y)}\ (a, b)$. The line through P which is $\perp$ both L and M is the vertical line $x = a$, and it meets L and M at the points $(a, 0)$ and (a, c) respectively. The distances from P to these points are $|b|$ and $|c - b|$ respectively, so the locus is defined by $b^2 = (c - b)^2$, or

$$0 = c(c - 2b)$$

which reduces to $c = 2b$, so that P lies on the locus $\Longleftrightarrow$ it lies on the line

$$y = \frac{c}{2}$$

which is horizontal & hence $\parallel$ both L & M. ∎

Clearly the second proof is much simpler.