Exercises for Unit II
(Vector algebra and Euclidean geometry)

II.1 : Approaches to Euclidean geometry

1. What is the minimum number of planes containing three concurrent , noncoplanar lines in coordinate $3-$space $\mathbb{R}^3$?

2. What is the minimum number of planes in coordinate $3-$space $\mathbb{R}^3$ containing five points, no four of which are coplanar? **[*Hint:*** No three of the points are collinear.**]**

3. Suppose that **a, b, c, d** are four noncoplanar points in $\mathbb{R}^3$. Explain why the lines **ab** and **cd** are disjoint but not coplanar (in other words, they form a pair of ***skew lines***).

4. Let **L** be a line in coordinate $2-$space $\mathbb{R}^2$ or $3-$space $\mathbb{R}^3$, let **x** be a point not on **L**, and let $\mathbf{p}_1, \ldots, \mathbf{p}_n$ be points on **L.** Prove that the lines $\mathbf{x}\mathbf{p}_1, \ldots, \mathbf{x}\mathbf{p}_n$ are distinct. Why does this imply that $\mathbb{R}^2$ and $\mathbb{R}^3$ contain infinitely many lines?

5. Suppose that $\mathbf{L}_1, \ldots, \mathbf{L}_n$ are lines in coordinate $2-$space $\mathbb{R}^2$ or $3-$space $\mathbb{R}^3$. Prove that there is a point **q** which does not lie on any of these lines. **[*Hint:*** Take a line **M** which is different from each of $\mathbf{L}_1, \ldots, \mathbf{L}_n$; for each j we know that **M** and $\mathbf{L}_j$ have at most one point in common, but we also know that **M** has infinitely many points.**]**

6. Suppose that $\mathbf{p}_1, \ldots, \mathbf{p}_n$ are points in coordinate $2-$space $\mathbb{R}^2$ or $3-$space $\mathbb{R}^3$. Prove that there is a line **L** which does not contain any of these points. **[*Hint:*** Let **x** be a point not equal to any of $\mathbf{p}_1, \ldots, \mathbf{p}_n$ and take a line **L** through **x** which is different from each of $\mathbf{x}\mathbf{p}_1, \ldots, \mathbf{x}\mathbf{p}_n$.**]**

7. Let **P** be a plane in coordinate $3-$space $\mathbb{R}^3$, let **a, b, c** be noncolinear points on **P,** let **z** be a point which is not on **P,** and let **u** and **v** be distinct points on the line **cz.** Show that the planes **P, abu** and **abv** are distinct. Using this and ideas from Exercise **4,** prove that there are infinitely many planes in coordinate $3-$space $\mathbb{R}^3$ which contain the line **ab.**

8. Suppose we have an abstract system $(\mathbf{P}, \mathcal{L})$ consisting of a set **P** whose elements we call **points** and a family of proper subsets $\mathcal{L}$ that we shall call **lines** such that the points and lines satisfy axioms $(\mathbf{I}-\mathbf{1})$ and $(\mathbf{I}-\mathbf{2})$. Assume further that every line **L** in **P** contains at least three points. Prove that **P** contains at least seven points.

Remark: The notes give an example of a system $(\mathbf{P}, \mathcal{L})$ such that that every line in **P** contains ***exactly*** three points and **P** itself contains ***exactly*** seven points.

II.2 : Synthetic axioms of order and separation

1. Suppose that **A, B, C** are collinear points in $\mathbb{R}^3$ whose coordinates are given by (a_1, a_2, a_3), (b_1, b_2, b_3), and (c_1, c_2, c_3) respectively. Prove that **A*B*C** holds if we have $a_1 < b_1 < c_1$, $a_2 = b_2 = c_2$, and $a_3 > b_3 > c_3$.

2. In the coordinate plane $\mathbb{R}^2$, let **A** $= (1, 0)$, **B** $= (0, 1)$, **C** $= (0, 0)$ and **D** $= (-2, -1)$. Show that the lines **AB** and **CD** intersect in a point **X** such that **A*X*B** and **D*C*X** hold. **[*Hint:*** Construct **X** explicitly.**]**

3. In the coordinate plane $\mathbb{R}^2$, suppose that **A, B, C, D, E** are points not on the same line such that **A*B*C** and **A*D*E** hold. Prove that the segments **(BE)** and **(CD)** have a point in common. **[*Hint:*** Use barycentric coordinates with respect to the points **A, B** and **D.]**

4. In the coordinate plane $\mathbb{R}^2$, let **A** $= (4, -2)$ and **B** $= (6, 8)$, and let **L** be the line defined by the equation $4y = x + 10$. Determine whether **A** and **B** lie on the same side of **L.**

5. In the coordinate plane $\mathbb{R}^2$, let **A** $= (8, 5)$ and **B** $= (-2, 4)$, and let **L** be the line defined by the equation $y = 3x - 7$. Determine whether **A** and **B** lie on the same side of **L.**

6. Let **L** be a line in the plane **P,** and suppose that **M** is some other line in **P** such that **L** and **M** have no points in common. Prove that all points of **M** lie on the same side of **L** in **P.**

7. State (in precise terms) and prove two generalizations of the previous result to disjoint planes in $3-$ space. One should involve disjoint planes in $3-$ space, and the other should involve a line **L** and a plane **P** that are disjoint.

8. Suppose that **L** is a line in the plane of triangle $\triangle$**ABC.** Prove that **L** cannot meet all three of the open sides **(AB), (BC)** and **(AC).**

9. Let $\mathbf{V}$ be a vector space over the real numbers, and let $\mathbf{S} = \{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a subset of $\mathbf{V}$. A vector $\mathbf{x}$ in $\mathbf{V}$ is said to be a ***convex combination*** of the vectors in $\mathbf{S}$ if $\mathbf{x}$ is a linear combination of the form $a_0\mathbf{v}_0 + a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n$ such that $a_0 + a_1 + \dots + a_n = 1$ and $0 \le a_j \le 1$ for all j.

Let $\mathbf{S} = \{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a subset of $\mathbf{V}$, and let $\mathbf{T} = \{\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_m\}$ be a set of vectors in $\mathbf{V}$ which are convex combinations of the vectors in $\mathbf{S}$. Suppose that $\mathbf{y}$ is a vector in $\mathbf{V}$ which is a convex combination of the vectors in $\mathbf{T}$. Prove that $\mathbf{y}$ is also a convex combination of the vectors in $\mathbf{S}$.

10. For which values of t is the point $(17, t)$ on the same side of the line defined by the equation $2x + 3y = 6$ as the point $(4, 5)$?

11. Given the three points $(2, 5)$, $(6, 5)$ and $(6, 2)$, find two of them which lie on the same side of the line defined by the equation $3y - 2x = 1$.

II.3 : Measurement axioms

1. Suppose we are given a line containing the two points **A** and **B.** Then every point **X** on the line can be expressed uniquely as a sum $\mathbf{A} + k(\mathbf{B} - \mathbf{A})$ for some real number k. Let $\mathbf{f} : \mathbf{L} \to \mathbb{R}$ be defined by $\mathbf{f}(\mathbf{X}) = k\,d(\mathbf{A}, \mathbf{B})$. Prove that **f** defines a $1-1$ correspondence such that $d(\mathbf{X}, \mathbf{Y}) = |\mathbf{f}(\mathbf{X}) - \mathbf{f}(\mathbf{Y})|$ for all points **X, Y** on **L.** **[*Hint:*** Recall that $d(\mathbf{X}, \mathbf{Y}) = |\mathbf{X} - \mathbf{Y}|$.**]**

2. Suppose that we are given a line **L** and **two** distinct $1-1$ correspondences between **L** and the real line $\mathbb{R}$ which satisfy the condition in the Ruler Postulate $(\mathrm{D}-3)$, say $\mathbf{f} : \mathbf{L} \to \mathbb{R}$ and $\mathbf{g} : \mathbf{L} \to \mathbb{R}$. Prove that these functions satisfy a relationship of the form $\mathbf{g}(\mathbf{X}) = a\,\mathbf{f}(\mathbf{X}) + b$, where a and b are real numbers with $a = \pm 1$. **[*Hint:*** Look at the function $\mathbf{h} = \mathbf{g} \circ \mathbf{f}^{-1}$, which is a distance preserving $1-1$ correspondence from $\mathbb{R}$ to itself. Show that such a map has the form $\mathbf{h}(t) = at + b$, where a and b are as above. To do this, first show that if **h** is a distance preserving $1-1$ correspondence from $\mathbb{R}$ to itself then so is $\mathbf{k}(t) = \mathbf{h}(t) - \mathbf{h}(0)$, then show that **k** must be multiplication by ± 1.**]**

3. In the coordinate plane, determine whether the point $\mathbf{X} = (9, 4)$ lies in the interior of $\angle\mathbf{ABC}$, where $\mathbf{A} = (7, 10)$, $\mathbf{B} = (2, 1)$ and $\mathbf{C} = (11, 1)$. Also, determine the values of k for which $(17, k)$ lies in the interior of $\angle\mathbf{ABC}$.

4. Answer the same questions as in the preceding exercise for $\mathbf{X} = (30, 200)$ and $\mathbf{X} = (75, 135)$.

5. Answer the same questions as in Exercise 3 when $\mathbf{X} = (-5, 8)$, $\mathbf{A} = (-4, 8)$, $\mathbf{B} = (-1, 2)$ and $\mathbf{C} = (-5, -12)$.

6. Suppose we are given $\angle\mathbf{ABC}$. Prove that the open segment **(AC)** is contained in the interior of $\angle\mathbf{ABC}$.

7. Suppose we are given $\triangle \mathbf{ABC}$ and a point **D** in the interior of this triangle, and let **E** be any point in the same plane except **D.** What general conclusion about the intersection of $\triangle \mathbf{ABC}$ and **[DE** seems to be true? Illustrate this conjecture with a rough sketch.

8. Suppose that we are given $\triangle \mathbf{ABC}$ and a point **X** on **(BC).** Prove that the open segment **(AX)** is contained in the interior of $\triangle \mathbf{ABC}$.

9. Let $\angle \mathbf{ABC}$ be given. Prove that there is a point **D** on the same plane such that **D** and **A** lie on opposite sides of **BC,** but $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{D}, \mathbf{B})$ and $|\angle \mathbf{ABC}| = |\angle \mathbf{DBC}|$. [***Hint:*** This uses both the Ruler and Protractor Postulates.]

10. Suppose we are given $\triangle \mathbf{ABC}$ and a point **D** in its interior. Prove that **D** lies on an open segment **(XY),** where **X** and **Y** lie on $\triangle \mathbf{ABC}$ and at least one of **X, Y** is not a vertex. [***Further question:*** Why is the converse also true?]

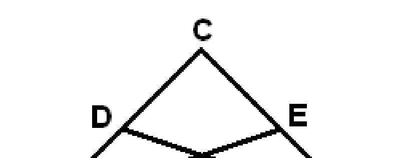
II.4 : Congruence, superposition and isometries

1. Let $\angle \mathbf{ABC}$ be given. Prove that there is a unique ***angle bisector*** ray **[BD** such that **(BD** is contained in the interior of $\angle \mathbf{ABC}$ and $|\angle \mathbf{ABD}| = |\angle \mathbf{DBC}| = \frac{1}{2}\,|\angle \mathbf{ABC}|$. [***Hints:*** Let **E** be the unique point on **(BA** such that $d(\mathbf{B}, \mathbf{E}) = d(\mathbf{B}, \mathbf{C})$, and let **D** be the midpoint of **[CE].** Recall that there should be proofs for both existence and uniqueness.]

2. Give an example of a triangle $\triangle \mathbf{ABC}$ for which the standard formal congruence statement $\triangle \mathbf{ABC} \cong \triangle \mathbf{BCA}$ is false.

3. Suppose we are given $\triangle \mathbf{ABC}$ and $\triangle \mathbf{DEF}$ such that $\triangle \mathbf{ABC} \cong \triangle \mathbf{DEF}$. Explain why we also have $\triangle \mathbf{ACB} \cong \triangle \mathbf{DFE}$ and $\triangle \mathbf{BCA} \cong \triangle \mathbf{EFD}$.

4. Suppose $\triangle \mathbf{ABC}$ is an isosceles triangle with $d(\mathbf{A}, \mathbf{C}) = d(\mathbf{B}, \mathbf{C})$, and let **D** and **E** denote the midpoints of **[AC]** and **[BC]** respectively. Prove that $\triangle \mathbf{DAB} \cong \triangle \mathbf{EBA}$.



5. Suppose we are given $\triangle \mathbf{ABC}$ and $\triangle \mathbf{DEF}$ such that $\triangle \mathbf{ABC} \cong \triangle \mathbf{DEF}$, and suppose that we have points **G** on **(BC)** and **H** on **(EF)** such that **[AG** and **[DH** bisect $\angle \mathbf{BAC}$ and $\angle \mathbf{EDF}$ respectively. Prove that $\triangle \mathbf{GAB} \cong \triangle \mathbf{HDE}$.

6. Conversely, in the setting of the previous exercise suppose that we are not given the condition $\triangle$**ABC** $\cong$ $\triangle$**DEF,** but we are given that $\triangle$**GAB** $\cong$ $\triangle$**HDE.** Prove that $\triangle$**ABC** $\cong$ $\triangle$**DEF.**

7. Let **K** be a convex subset of $\mathbb{R}^n$. A point **X** in **K** is said to be an ***extreme point*** of **K** if it is not between two other points of **K.** Suppose that **T** is an affine transformation of $\mathbb{R}^n$, and suppose that **T** maps the convex set **K** onto the convex set **L.** Prove that **T** maps the extreme points of **K** to the extreme points of **L.**

8. Suppose that **T** is the affine transformation of $\mathbb{R}^n$ given by **T(x) = L(x) + v,** where **L** is an invertible linear transformation and **v** is a fixed vector in $\mathbb{R}^n$. If **L** is given by the $n \times n$ matrix **A** $= (a_{i,j})$ and **v** $= (b_1, \ldots, b_n)$, then one has the following expression for **y = T(x)** in terms of coordinates:

$$y_i = a_{i,1}\,x_1 + \ldots + a_{i,n}\,x_n + b_i$$

Let **DT** be the **<u>derivative matrix</u>** whose (i,j) entry is given by

$$\frac{\partial\, y_i}{\partial\, x_j}$$

(a) Show that the (i,j) entry of **DT** is equal to $a_{i,j}$.

(b) If $\mathbf{T}_1$ and $\mathbf{T}_2$ are affine transformations of $\mathbb{R}^n$, explain why $\mathbf{D}(\mathbf{T}_1 \circ \mathbf{T}_2)$ is the matrix product $\mathbf{D}(\mathbf{T}_1)\,\mathbf{D}(\mathbf{T}_2)$. **[*<u>Hint:</u>*** Expand the composite $\mathbf{T}_1 \circ \mathbf{T}_2$.**]**

(c) Explain why **T** is a translation if and only if **D(T)** is the identity matrix.

(d) If **T** is a translation and **S** is an arbitrary affine transformation, prove that the composite $\mathbf{S}^{-1} \circ \mathbf{T} \circ \mathbf{S}$ is a translation. What is its value at the vector $\mathbf{0}$?

9. The ***vertical reflection*** $\mathbf{S}(c)$ about the horizontal line $y = c$ in $\mathbb{R}^2$ is the affine map defined by $(x_1, x_2) = (x_1, 2c - x_2)$. Show that $\mathbf{S}(c)$ sends the horizontal line $y = c$ into itself and interchanges the horizontal lines $y = 2c$ and $y = 0$. Prove that the composite of two vertical reflections $\mathbf{S}(a)\mathbf{S}(b)$ is a translation, and the composite of three vertical reflections $\mathbf{S}(a)\mathbf{S}(b)\mathbf{S}(c)$ is a vertical reflection $\mathbf{S}(d)$; evaluate d explicitly. **[*<u>Hints:</u>*** For the twofold composite, what is the derivative matrix? Also, explain why the twofold composite $\mathbf{S}(a)\mathbf{S}(b)$ sends $(0, 0)$ to a point whose first coordinate is equal to zero.**]**

10. Let **A** be the orthogonal matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

where θ is a real number. Show that there is an orthonormal basis $\{\mathbf{u}, \mathbf{v}\}$ for $\mathbb{R}^2$ such that $\mathbf{Au} = \mathbf{u}$ and $\mathbf{Av} = -\mathbf{v}$.

11. Let **A** be the orthogonal rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where θ is a real number which is ***not*** an integral multiple of 2π, let $\mathbf{b} \in \mathbb{R}^2$, and let **T** be the Galilean transformation $\mathbf{T(x)} = \mathbf{Ax} + \mathbf{b}$. Prove that there is a unique vector **z** such that $\mathbf{T(z)} = \mathbf{z}$. [***Hint:*** **A** is not the identity matrix, and in fact $\mathbf{A} - \mathbf{I}$ is invertible; prove the latter assertion.]

II.5 : Euclidean parallelism

1. Suppose that **L** and **M** are skew lines in $\mathbb{R}^3$. Prove that there is a unique plane **P** such that **L** is contained in **P** and **M** is parallel to (*i.e.*, disjoint from) **P**. [***Hints:*** Write **L** and **M** as $\mathbf{x} + \mathbf{V}$ and $\mathbf{y} + \mathbf{W}$, where **V** and **W** are $1-$dimensional vector subspaces. Since **L** and **M** are not parallel, we know that **V** and **W** are distinct. Let **U** be the vector subspace $\mathbf{V} + \mathbf{W}$. Why is **U** a $2-$dimensional subspace? Set $\mathbf{P} = \mathbf{x} + \mathbf{U}$ and verify that **P** has the desired properties; in particular, if **M** and **P** have a point **z** in common, note that $\mathbf{M} = \mathbf{z} + \mathbf{W}$ and $\mathbf{P} = \mathbf{z} + \mathbf{U}$.]

2. Suppose that **P** and **Q** are parallel planes in $\mathbb{R}^3$, and let **S** be a plane which meets each of them in a line. Prove that the lines of intersection $\mathbf{S} \cap \mathbf{P}$ and $\mathbf{S} \cap \mathbf{Q}$ must be parallel.

3. Suppose that we are given three distinct planes **S**, **P** and **Q** in $\mathbb{R}^3$ such that **S** is parallel to both **P** and **Q**. Prove that **P** and **Q** are parallel. [***Hint:*** Through a given point **x** not on **S**, how many planes are there that pass through **x** and are parallel to **S**?]

4. Suppose that **v** and **w** are linearly independent vectors in $\mathbb{R}^3$, and let **z** be a third vector in $\mathbb{R}^3$. Let a and b be real numbers. Prove that the function $\mathbf{f}(a, b) = |\mathbf{z} - a\mathbf{v} - b\mathbf{w}|$ takes a minimum value when $\mathbf{z} - a\mathbf{v} - b\mathbf{w}$ is perpendicular to both **v** and **w**. [***Hint:*** Take an orthonormal basis **e, f** for the span of **v** and **w**, and let $\mathbf{Q(z)} = \mathbf{z} - \langle \mathbf{z}, \mathbf{e} \rangle \mathbf{e} - \langle \mathbf{z}, \mathbf{f} \rangle \mathbf{f}$. Why is **z** perpendicular to **e, f, v,** and **w**? Find the length squared of the vector

$$\mathbf{z} - s\mathbf{e} - t\mathbf{f} = \mathbf{Q(z)} - [s - \langle \mathbf{z}, \mathbf{e} \rangle]\mathbf{e} - [t - \langle \mathbf{z}, \mathbf{f} \rangle]\mathbf{f}$$

using the fact that **e, f** and **Q(z)** are mutually perpendicular. Why is the (square of the) length minimized when the coefficients of **e** and **f** are zero? Why is the minimum value of $\mathbf{g}(s, t) = |\mathbf{z} - s\mathbf{e} - t\mathbf{f}|$ equal to the minimum value of $\mathbf{f}(a, b) = |\mathbf{z} - a\mathbf{v} - b\mathbf{w}|$?]

5. Suppose that we have two skew lines in $\mathbb{R}^3$ of the form **0a** and **bc.** Let **x** and **y** be points of **0a** and **bc** respectively. Prove that the distance $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$ is minimized when $\mathbf{x} - \mathbf{y}$ is perpendicular to **a** and $\mathbf{c} - \mathbf{b}$. (In other words, ***the shortest distance between the two skew lines is along their common perpendicular.***)

6. Let **W, X, Y, Z** be four points in $\mathbb{R}^3$, no three of which are collinear, let **A, B, C, D** be the midpoints of **WX, XY, YZ,** and **ZW,** and suppose that we have $\mathbf{AB} \neq \mathbf{CD}$ and $\mathbf{AD} \neq \mathbf{BC}$. Prove that **AB || CD** and **AD || BC**.

7. Let $\angle$**ABC** be given, and let **D** lie in the interior of $\angle$**ABC.** Prove that **D** lies on an open segment **(XY),** where **X** $\in$ **(BA** and **Y** $\in$ **(BC.** [***Hint:*** The proof requires the use of Playfair's Axiom. In Unit **V** there is an exercise which shows that result is not necessarily true if Playfair's axiom does not hold.]