Math 133
Winter 2013

MORE EXERCISES FOR SECTIONS II.1 AND II.2

There are drawings on the next two pages to accompany the starred $(*)$ exercises.

B1. Let L be a line in $\mathbb{R}^3$, and let $\mathbf{x}$ be a point which does not lie on L. Using the Incidence Axioms, prove that there is a unique plane P such that $\mathbf{x} \in P$ and $L \subset P$. [*Hint:* If A and B are two points of L, why are A, B and X noncollinear?.]

B2. $(*)$ Suppose that we are given three distinct lines L, M, N in $\mathbb{R}^3$ such that (i) the lines all contain some point X, (ii) each of the lines has a point in common with a fourth line K which does not contain X. Prove that there is a plane containing all four lines.

B3. Suppose that we are given points A, B, C and X, Y, Z such that $A*B*C$ and $X*Y*Z$ both hold, and in addition we have $d(A,C) = d(X,Z)$ and $d(B,C) = d(Y,Z)$. Prove that $d(A,B) = d(X,Y)$. [*If equals are subtracted from equals, the differences are equal.*]

B4. $(*)$ Suppose that we are given points A, B, C such that $A * B * C$. Prove that (AB) is a proper subset of (AC).

B5. $(*)$ Suppose that $A \neq B$; if AB is the line joining A to B, prove that $AB = [AB \cup [BA$.

B6. $(*)$ Suppose that $A * B * C$; prove that $[AB = [AB] \cup [BC$.

B7. Let L be a line in $\mathbb{R}^2$, and let M be a second line in $\mathbb{R}^2$ such that L and M meet at the point A.

- (a) $(*)$ If X and Y are points of M such that $A * X * Y$ is true, prove that X and Y lie on the same side of L.
- (b) If X and Y are points of M such that $X * A * Y$, prove that X and Y lie on opposite sides of L. [*Hint:* For both parts of this problem, show that the alternatives are impossible.]

B8. $(*)$ Suppose that we are given distinct points A, B in $\mathbb{R}^2$, and suppose also that C and D lie on opposite sides of the line AB. Prove that $[AC$ and $[BD$ have no points in common.

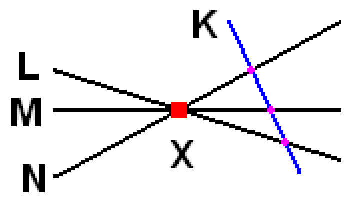
B9. $(*)$ (i) Suppose that we are given three noncollinear points A, B, C in $\mathbb{R}^2$. Prove that $\Delta ABC \cap AB = [AB]$. [*Hint:* If X is a point on $[BC]$ or $[AC]$ other than A or B, explain why X cannot lie on AB.]

B10. For each of the choices below, determine whether X and Y lie on the same side as the line L defined by the corresponding equation.

- (a) $X = (3,6)$, $Y = (1,7)$, and L is defined by the equation $9x - 4y = 7$.
- (b) $X = (8,5)$, $Y = (-2,4)$, and L is defined by the equation $y = 3x - 7$.
- (c) $X = (7,-6)$, $Y = (4,-8)$, and L is defined by the equation $2x + 3y + 5 = 0$.
- (d) $X = (0,1)$, $Y = (-2,6)$, and L is defined by the equation $3y = 2 - 7x$.

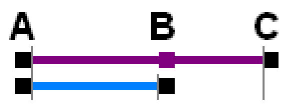
DRAWINGS TO ACCOMPANY ADDITIONAL EXERCISES, SET B

B2.



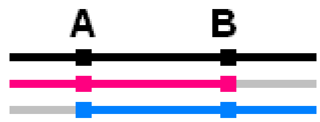
The figure suggests that **L, M,** and **N** are all contained in the plane containing the line **K** and the point **X,** which does not lie on **K.**

B4.



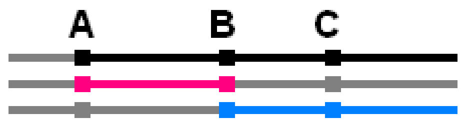
The goals is to show that (i) if **X** is between **A** and **B,** then **X** is also between **A** and **C,** (ii) there is some point **Y** such that **Y** is between **A** and **C** but **Y** is not between **A** and **B.**

B5.



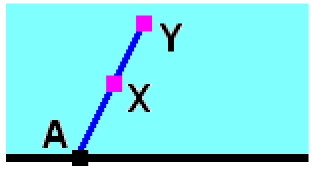
The goal is to show that the line **AB** is the union of the overlapping rays colored pink and light blue**.**

B6.



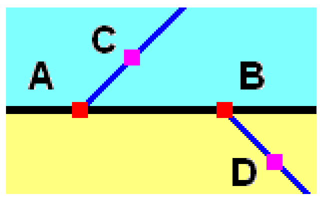
The goal is to show that the ray **[AB** is the union of the pink segment **[AB]** and the light blue ray **[BC.**

B7(*a*).



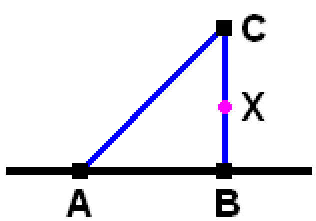
The goal is to show that **X** and **Y** lie on the same side of **L,** or equivalently that the other alternatives are impossible**.** These alternatives are that one of **X, Y** lies on **L** or that **X** and **Y** lie on opposite sides of **L.**

B8.



The points **B** and **D** are assumed to lie on opposite sides of the line **AB,** and the objective is to prove that the rays **[AC** and **[BD** have no points in common**.**

B9.



To show that the intersection of triangle **ABC** and line **AB** is the segment **[AB],** it will suffice to show that if **X** is a point of the triangle on either **(BC]** or **(AC]** then **X** does not lie on the line **AB.**

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Fall 2013

EXERCISES INVOLVING CONVEX FUNCTIONS

Some of these exercises show that certain subsets of the coordinate plane which *appear* to be convex actually satisfy the mathematical condition defining convexity. Others illustrate how some arguments in `convex-functions.pdf` can be modified to yield further results on the topic.

K0. Show that if K is an open convex set and f is a convex function on K then the *open epigraph* consisting of all $(\mathbf{x}, u) \in K \times \mathbb{R}$ such that $u > f(\mathbf{x})$ (*i.e.*, we have strict inequality) is also a convex set. [*Hint:* Imitate the relevant portion of the proof for Theorem 1 in `convex-functions.pdf`.]

K1. Using the Second Derivative Test, verify that the following functions are convex on the indicated subsets of the real line:

(i) The hyperbolic cosine function $f(x) = \cosh(x) = \frac{1}{2}(e^x + e^{-x})$, for all real numbers x.

(ii) The hyperbolic sine function $f(x) = \sinh(x) = \frac{1}{2}(e^x - e^{-x})$, for all **positive** real numbers x.

(iii) The function $f(x) = \tan x$, for all x such that $0 \le x < \frac{1}{2}\pi$.

(iv) The function $f(x) = \sin x$, for all x such that $\pi \le x \le 2\pi$.

(v) The **negative** of the natural logarithm function $f(x) = -\log_e x$, for all positive real numbers x.

K2. A real valued function f on a convex set is said to be concave (= *concave downward* in calculus textbook terminology) if its negative $-f$ is convex.

(i) Show that a function f is concave if and only if the set $\{(x, y) \mid y \le f(x)\}$ is convex. [*Hint:* Recall that $-f$ is convex.]

(ii) Show that if f is defined on an interval in the real line and has a continuous second derivative with $f'' \le 0$, then f is concave.

K3. Suppose that f and g are real valued functions defined on the same interval J in the real line. If f is convex, g is concave, and $f \le g$ on J, prove that the set $\{(x, y) \mid x \in J, f(x) \le y \le g(x)\}$ is convex. [*Hint:* The intersection of two convex sets is convex.]

K4. Let a be a positive real number. Using the preceding exercise, prove that the solid elliptical region defined by

$$\left\{ (x, y) \;\middle|\; |x| \le a , \quad |y| \le \sqrt{1 - \frac{x^2}{a^2}} \right\}$$

is convex. [*Hint:* If $f(x)$ is the function at the end of the preceding display, explain why the solid elliptical region is defined by the inequalities $-a \le x \le a$ and $-f(x) \le y \le f(x)$.]

K5. A real valued function f on a convex set K is said to be *strictly convex* if for all $\mathbf{x}, \mathbf{y} \in K$ and $t \in (0,1)$ we have a strict inequality

$$f\,(t\,\mathbf{x} \;+\; (1-t)\,\mathbf{y}\,) \quad < \quad t\,f(\mathbf{x}) \;+\; (1-t)\,f(\mathbf{y})\ .$$

(i) Show that if f is defined on an interval in the real line and has a continuous second derivative with $f'' > 0$, then f is strictly convex. [*Hint:* As in the proof of Theorem 2 in `convex-functions.pdf`, assume first that f is defined on a closed interval $[a,b]$ let g be the linear function such that $f(a) = g(a)$ and $f(b) = g(b)$. Using the methods in the arguments for Theorem 2 and Lemma 3, prove that the stronger condition $f'' > 0$ implies that $g(x) - f(x)$ is strictly positive for all x satisfying $a < x < b$. Conclude by showing this yields strict convexity.]

(ii) Use the preceding to show that a function satisfying the hypotheses in Theorem 5 of `convex-functions.pdf` is strictly convex.

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Winter 2009

MORE EXERCISES FOR SECTIONS II.3 AND II.4

All of the supplementary exercises for Section I.4 and many of those for Sections II.1–II.2 were fairly elementary and intended to help reinforce basic concepts. In contrast, the exercises below are more challenging and included to cover some additional material.

C1. Suppose that A, B, C are noncollinear points in a plane P, and let $X \in \Delta ABC$. Prove that if X is a vertex of ΔABC then there are (at least) two lines L and M such that X lies on both and each contains at least three distinct points of ΔABC, but if X is not a vertex then there is only one line L in P such that $X \in L$ and L contains at least three points of ΔABC. [*Hint:* The conclusion in Exercise II.2.8 is useful for establishing part of this result.]

C2. Suppose that we are given ΔABC and ΔDEF in a plane P such that $\Delta ABC = \Delta DEF$. Prove that $\{A, B, C\} = \{D, E, F\}$. [*Hint:* Use the preceding exercise.]

C3. Suppose that we are given a triangle ΔABC in a plane P, and suppose that L is a line in P such that L contains a point X in the interior of ΔABC. Prove that L and ΔABC have (at least) two points in common.

C4. [*In this exercise we shall view points of* $\mathbb{R}^n$ *as* $n \times 1$ *column vectors and identify scalars with* 1×1 *matrices in the obvious fashion.*] Let T be an affine transformation of $\mathbb{R}^3$ and write it as $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, where A is an invertible 3×3 matrix and $\mathbf{b}$ is some vector in $\mathbb{R}^3$. Let P be the plane defined by the equation $C\mathbf{x} = d$, where C is a 3×1 matrix and $d \in \mathbb{R}$, and let Q be the image of P; in other words, Q is the set of all vectors $\mathbf{y}$ such that $\mathbf{y} = T(\mathbf{x})$ for some $\mathbf{x} \in P$. Prove that Q is also a plane, and give an explicit equation of the form $U\mathbf{y} = v$ (where U is 1×3 and v is a scalar) which defines Q. [*Hint:* Solve $T(\mathbf{x}) = \mathbf{y}$ for $\mathbf{x}$ in terms of $\mathbf{y}$, A and $\mathbf{b}$.]

NOTE. One specific consequence of the preceding result is that planar figures (which are contained in some plane) and nonplanar figures (which are not contained in any plane) cannot be congruent to each other.

C5. Suppose that we are given coplanar points A, B, C, D such that no three are collinear, and suppose further that the following conditions hold:

(1) B does not lie in the interior of $\angle CAD$.

(2) C does not lie in the interior of $\angle BAD$.

(3) D does not lie in the interior of $\angle CAB$.

Prove that $|\angle BAC| + |\angle CAD| + |\angle DAB| = 360°$. [*Hint:* Show that (CD) meets AB in a point E such that $E * A * B$ holds.]