Math 133
Fall 2013

SUPPLEMENTARY EXERCISES ON AFFINE EQUIVALENCE

T1. Let K be a nonempty convex subset of $\mathbb{R}^2$, and let L be a line in $\mathbb{R}^2$ such that $K \cap L$ is empty. Prove that all the points of K lie on the same side of L. [*Hint:* Assume that K contains points on both sides of L and derive a contradiction.]

T2. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be noncollinear points in $\mathbb{R}^2$, and let F be an affine transformaton of $\mathbb{R}^2$. Prove that F sends $\angle\mathbf{abc}$ to $\angle F(\mathbf{a})F(\mathbf{b})F(\mathbf{c})$. [*Hint:* By Corollary II.4.8 F maps rays to rays, and we also know that if X and Y are subsets of $\mathbb{R}^2$ then F maps the union $X \cup Y$ to $F[X] \cup F[Y]$.]

T3. Assume the setting of the preceding exercise, and prove that F maps the exterior of $\angle\mathbf{abc}$ to the exterior of $\angle F(\mathbf{a})F(\mathbf{b})F(\mathbf{c})$. [*Hint:* By Theorem 12 in `affine-convex.pdf` and a previous exercise we know that F maps the angle and its interior to themselves.]

T4. If L and M are parallel lines in $\mathbb{R}^2$, then by Exercise **T1** we know that all points of L lie on the same side of M and all points of M lie on the same side of L. Define the *strip between L and M* to be the set of points $\mathbf{x}$ such that $\mathbf{x}$ and L are the same side of M and $\mathbf{x}$ and M are the same side of L.

(i) Prove that the strip between L and M is convex and nonempty. Specifically, prove that if $A \in L$ and $B \in M$, then the midpoint C of (AB) lies in this set.

(ii) Prove that if F is an affine transformation of $\mathbb{R}^2$ and L and M are parallel lines in $\mathbb{R}^2$, then F maps the strip between L and M to the strip between $F[L]$ and $F[M]$. [*Hint:* Apply Theorem 11 in `affine-convex.pdf`.]

T5. Using the idea for the solution to Exercise **T1**, prove the following uniqueness result for the half-planes in the Plane Separation Postulate:

> *Let L be a line in $\mathbb{R}^2$, and suppose that $\{H_1, H_2\}$ and $\{K_1, K_2\}$ are subsets of $\mathbb{R}^2 - L$ which satisfy the conditions in the Plane Separation Postulate. Namely, each subset is nonempty and convex, we have $H_1 \cap H_2 = K_1 \cap K_2 = \emptyset$, $H_1 \cup H_2 = K_1 \cup K_2 = \mathbb{R}^2 - L$, and for $C = H or K$, if $\mathbf{x} \in C_1$ and $\mathbf{y} \in C_2$, then $(\mathbf{xy}) \cap L \neq \emptyset$.* THEN *either $H_1 = K_1$ and $H_2 = K_2$ or else $H_1 = K_2$ and $H_2 = K_1$.*