The following pages give a hypothetical physical interpretation of the Poincaré unit disk model for hyperbolic geometry. This passage is taken from the following classic book:

W. W. Sawyer, ***Prelude to Mathematics*** (Revised edition of the 1955 original). Dover Publications, Mineola, NY, 2011.

lives. The two theories are not different in their essential content; only in picturesque details. It is a matter purely of taste which picture one prefers.

To illustrate this point I will give you a 'flat' model of a two-dimensional universe embodying 'Possibility III'. This universe could also be got by considering objects sliding about on a suitable curved surface. The model here used is due to Poincaré (1854–1912).

POINCARÉ'S UNIVERSE

This universe is contained in the interior of a circle. At the centre of the circle the temperature is fairly high, but as you go away from the centre the temperature falls, and reaches the absolute zero at the circumference. The law for the temperature is quite a simple one. If a is the radius of the circle, at a distance r from the centre of the circle, the temperature is $T = a^2 - r^2$.

If now any object moves about in this universe, its size is affected by the variation in temperature. We shall suppose that the length of any object varies in proportion to the temperature T. At the circumference of the circle, where $r = a$ and $T = 0$, the length of the object will shrink to zero. The breadth varies in exactly the same way as the length.

But the inhabitants of this universe are not aware of the temperature. We suppose them to have no nerves sensitive to heat, so they do not feel temperature directly. Nor can they measure it by means of thermometers. The ordinary thermometer depends on the fact that mercury expands more rapidly than glass. But in this universe, every object expands and contracts in exactly the same way. If a creature is six feet long when it is at the centre of the circle, and it goes to a colder part, it will still find itself six feet long. If its length seems to us to be halved, so also is the length of the foot ruler with which it measures itself. It is still six times as long as the ruler.

But surely, someone may object, it will notice the fact that objects become of zero size on the boundary? The answer to this is that no one can reach the boundary (Figure 22). For as a creature starts to walk towards the boundary, its size (as seen by us) decreases; the nearer it comes to the boundary, the more rapidly it shrinks. To itself it seems to be taking steps of equal size; to us it seems to take shorter and shorter steps, the law being such that, however many steps it may take, it will never reach the boundary.[1] The boundary, from the creature's point of view, is infinitely distant. Lines which meet there are parallel. Thus, although this universe for us occupies a finite space, for the people in it, it is infinite.

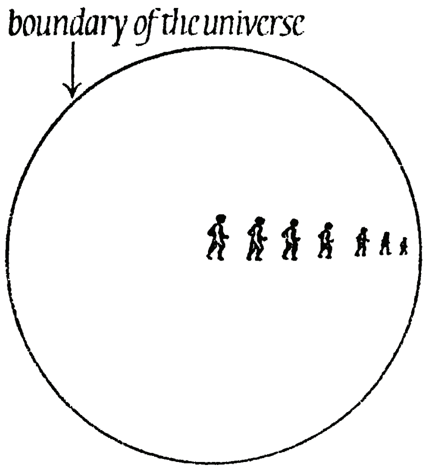

Figure 22

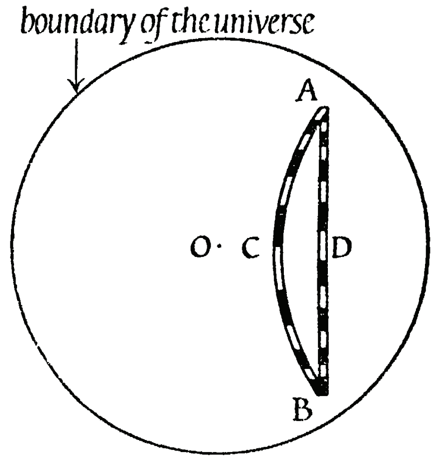

Figure 23

What will be a 'straight line' in such a universe? By definition, a 'straight line' is the shortest way from one point to another. The length of a path, for this purpose, is the length as measured by the creatures themselves. In Figure 23, *ADB* is a line that *we* should call straight. The light and dark segments of *ADB* show fifteen links of a chain connecting *A* and *B*. The links of this chain are – for the creatures – of equal length. The links near *D* are of course nearer the centre *O*; if these links were taken out to the

1. This can be seen by simple calculus. A length ds at the centre of the circle will, when transported to a distance r from the centre, occupy a space

$$dr = \frac{a^2 - r^2}{a^2} ds.$$

So the creature regards a distance dr (in our measurement) as having a length

$$ds = \frac{a^2}{a^2 - r^2} dr.$$

The creature's estimate of the distance from the centre to the edge of the circle is

$$\int_0^a \frac{a^2\,dr}{a^2 - r^2},$$

which is infinite.

colder parts near *A* and *B* they would look just like the links at present near *A* and *B*.

Another chain is shown in the path *ACB*. The links of this chain are the same size as those of the first chain. But as *C* is in a warmer position than *D*, there are fewer links in the chain *ACB* than in the chain *ADC*. *ACB* is a 'shorter' path than *ADC*. As a matter of fact *ACB* is the 'straight line' joining *A* to *B*. Fewer links are needed for the chain *ACB* than for any alternative route.

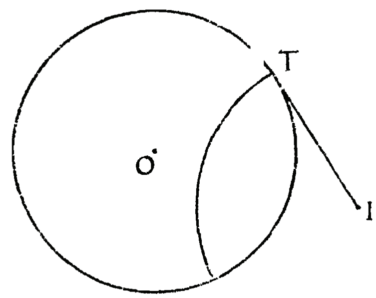

Figure 24

There is a simple geometrical construction for a 'straight line'. 'Straight lines' are in fact circles. Not any circle will do; it must be a circle that crosses the boundary circle at right angles (Figure 24). If *I* is any point outside the boundary circle, and *IT* is the tangent from *I* to the boundary circle, then the circle centre *I*, radius *IT* will do as a 'straight line'. One and only one such 'straight line' can be drawn to join any two points *A* and *B* in the universe. A stretched wire would naturally come to this shape; a ray of light would follow the same path. So, in the figure with the two chains, *ACB* is the shape that a chain stretched between *A* and *B* would take; the creatures could check this by eye; since light follows the track *ACB*, to a creature looking from *B* the point *C* would hide the point *A* – and this is the way we check that three points *A, C, B* are in line. So the shortest paths would look straight to the creatures inside the universe, although they do not look straight to us.

When the diagram illustrating Possibility III was drawn earlier, I remarked that it did not look right to us. Here is the same diagram with the same lettering, as it would be in Poincaré's universe. *AO* is the 'line' perpendicular to the 'line' *CD*. *P* is to the right of *O*, *Q* to the left. The dotted 'line' *NAM* is perpendicular to *OA*. As *P* moves to 'infinity' (i.e. to the boundary of the universe) the 'line' *AP* approaches the position *AR*.

This leaves a considerable angle between *AR* and *AM*. As *Q* moves to the left to 'infinity', the 'line' *AQ* approaches *AL*. Again, there is a considerable angle between *AL* and *AN*. If we take any direction from *A* that lies within the angle *LAN* or within the angle *RAM*, and proceed in a 'straight line' passing through *A* in this direction, we shall obtain a 'straight' path that does not meet *CD*. There are thus an infinity of 'lines' through *A* that do not meet *CD*.

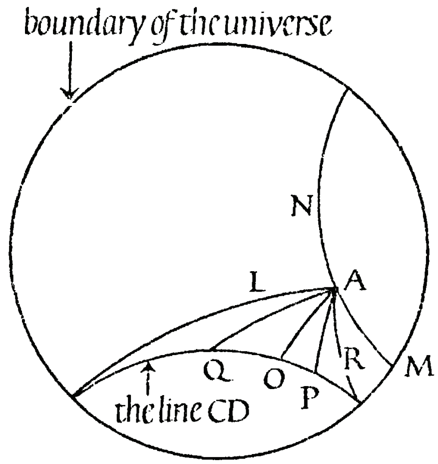

Figure 25

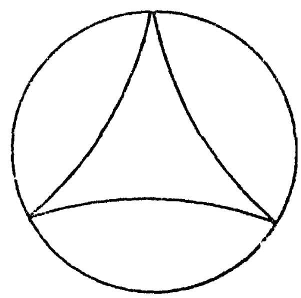
Figure 26

If we were to take a very small part of this universe, the temperature would vary very little within this small part. The geometry of a small part of this universe would accordingly be the same as Euclid's. The inhabitants of this universe, if their opportunities for travel were restricted, might well believe that they lived in a Euclidean plane. There is a moral in this for us, as we live within the solar system, which is a tiny speck in comparison with the distances between the stars.

In Poincaré's universe, the sum of the angles of a triangle is almost 180° for a small triangle, but the larger the triangle, the smaller the sum of the angles. If you look for instance at the triangle in Figure 25 formed by *AL*, *AR* and *CD*, the two angles on the boundary are nothing. The angle *LAR* is considerably less than 180°, and so the sum of the angles is less than 180°. In the triangle drawn in Figure 26 the sum of the angles is 0°.

By measuring the angles of a large triangle, it would be possible to discover whether one lived in a universe using Possibility I, II, or III.