PLANE SEPARATION AND VECTOR ALGEBRA

This is a more detailed look at the interpretation of plane separation in terms of coordinates. We shall verify in detail that *for each line L the set of points not on L satisfies the conditions in the Plane Separation Postulate.*

If L is a line in the coordinate plane $\mathbb{R}^2$, then L is defined by an equation of the form

$$0 \quad = \quad g(x,y) \quad = \quad Ax \; + \; By \; + \; C$$

where at least on of A, B is nonzero. The two half-planes determined by L are the sets where $g(x,y) > 0$ and $g(x,y) < 0$. We shall denote these half-planes (or sides of the line L) by H_1 and H_2 respectively.

The first thing to notice is that H_1 and H_2 are both nonempty. For each scalar k, consider the point $V_k = (kA, kB)$. We then have $g(x,y) = k(A^2+B^2)+C$, and since at least one of A, B is nonzero it follows that the coefficient $A^2 + B^2$ is positive. Therefore we can say that $g(V_k) = g(kA, kB)$ will be positive if $k > -C/(A^2 + B^2)$ and $g(V_k) = g(kA, kB)$ will be negative if $k < -C/(A^2 + B^2)$. Since there are infinitely values of k satisfying either of these inequalities, it follows that in fact both H_1 and H_2 contain infinitely many points.

We also need to check that H_1 and H_2 are both convex; in other words, if $P = (x,y)$ and $Q = (u,v)$ belong to one of these half-planes and $0 < t < 1$, then the point $P+t(Q-P)$ also belongs to the same half-plane. The key to this is the following chain of identities:

$$g\big(\,P{+}t(Q{-}P)\,\big) \quad = \quad g\big(\,x{+}t(u{-}x), y{+}t(v{-}y)\,\big) \quad = \quad A\big(\,x{+}t(u{-}x)\,\big) + B\big(\,y{+}t(v{-}y)\,\big) \quad =$$

$$(1-t)(Ax+By) \; + \; t(Ax+By) \; + \; C \quad = \quad (1-t)\cdot g(P) \; + \; t\cdot g(Q) \; .$$

If P and Q lie on the same side of L, then either $g(P)$ and $g(Q)$ are both positive or they are both negative. Note that t and $1-t$ are both positive in either case. If $g(P)$ and $g(Q)$ are positive, then it follows that

$$g\big(\,P + t(Q-P)\,\big) \quad = \quad (1-t)\cdot g(P) \; + \; t\cdot g(Q)$$

must also be positive since it is a sum of two products of positive numbers, while if $g(P)$ and $g(Q)$ are negative, then it follows that the expression is a sum of two products, each with one positive and one negative factor, and hence in this case $g\big(\,P + t(Q-P)\,\big)$ must be negative.

Finally, we need to show if P is in one half-plane and Q is in the other, then the open segment (PQ) and the line L have a point in common. In the terms of the preceding discussions, this means that we can find some t such that $0 < t < 1$ and $g\big(\,P{+}t(Q{-}P)\,\big) = 0$.

We shall only consider the case where $g(P) < 0 < g(Q)$; the other case, in which $g(P) > 0 > g(Q)$, can be obtained by interchanging the roles of P and Q in the argument below. By the fundamental identity displayed above, we need to find a value of t such that

$$0 \quad = \quad (1-t)g(P) \; + \; tg(Q) \quad = \quad g(P) \; + \; t\big(g(Q) - g(P) \big) \; .$$

The solution to this equation is

$$t \quad = \quad \frac{-g(P)}{g(Q) - g(P)}$$

where the denominator is positive since $g(Q) > g(P)$. By assumption $g(P)$ is negative, and therefore the entire expression for t is positive. Furthermore, we also have $0 < -g(P) < g(Q) - g(P)$, so it also follows that $t < 1$. Therefore, if we take t as given above, then the point $P + t(Q - P)$ will lie on both the open segment (PQ) and the line L.■