An alternate approach to similarity

The following generalities about similarity reflect the treatment in the lectures, which is not quite the same as the treatment in the course notes. We shall refer to the latter for some material.

There are four particularly important things to notice:

(1) For each $k > 0$, the transformation T_k of $\mathbb{R}^n$ which sends a vector $\mathbf{v}$ to $k\,\mathbf{v}$ is a similarity transformation with ratio of similitude k.

(2) The map T_k in the preceding item preserves angle measures.

(3) More generally, **every** abstract similarity preserves angle measures.

(4) Given ΔABC in $\mathbb{R}^n$ $(n = 2, 3)$ and $k > 0$, the image of ΔABC under T_k is a triangle ΔDEF such that $\Delta ABC \sim \Delta DEF$ with ratio of similitude k.

We shall first derive (1). If $\mathbf{x}$ and $\mathbf{y}$ are points in $\mathbb{R}^n$, then $|T_k(\mathbf{x}) - T_k(\mathbf{y})| = |k\,\mathbf{x} - k\,\mathbf{y}| = |k\,(\mathbf{x} - \mathbf{y})| = |k| \cdot |\mathbf{x} - \mathbf{y}|$, and since k is positive the last expression is equal to $k \cdot |\mathbf{x} - \mathbf{y}|$.$\blacksquare$

Note that T_k as defined above is an affine transformation, and hence a triple of points in $\mathbb{R}^n$ is collinear if and only if its image under T_k, and similarly for triples of noncollinear points, sets of four coplanar points, and sets of four noncoplanar points.

The derivation of (2) is somewhat longer. Since an angle measurement is entirely determined by its cosine, it will suffice to show that if $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are noncollinear, then

$$\cos \angle \mathbf{xyz} \quad = \quad \cos \angle T_k(\mathbf{x}) T_k(\mathbf{y}) T_k(\mathbf{z}) \ .$$

By definition, the right hand side of this equation is equal to

$$\frac{\langle T_k(\mathbf{x}) - T_k(\mathbf{y}), T_k(\mathbf{z}) - T_k(\mathbf{y}) \rangle}{|T_k(\mathbf{x}) - T_k(\mathbf{y})| \cdot |T_k(\mathbf{z}) - T_k(\mathbf{y})|} \quad = \quad \frac{\langle k\,\mathbf{x} - k\,\mathbf{y}, k\,\mathbf{z} - k\,\mathbf{y} \rangle}{|k\,\mathbf{x} - k\,\mathbf{y}| \cdot |k\,\mathbf{z} - k\,\mathbf{y}|}$$

and the right hand side may be simplified further to

$$\frac{\langle k(\mathbf{x} - \mathbf{y}), k(\mathbf{z} - \mathbf{y}) \rangle}{|k(\mathbf{x} - \mathbf{y})| \cdot |k(\mathbf{z} - \mathbf{y})|} \quad = \quad \frac{k^2 \cdot \langle \mathbf{x} - \mathbf{y}, \mathbf{z} - \mathbf{y} \rangle}{k^2 \cdot |\mathbf{x} - \mathbf{y}| \cdot |\mathbf{z} - \mathbf{y}|}$$

and if we cancel the k^2 factors in the numerator and denominator we obtain the standard formula for $\cos \angle \mathbf{xyz}$.$\blacksquare$

To derive (3), suppose that T is an abstract similarity transformation with ratio of similitude k. Then by Proposition III.5.1 and item (1) the map $S(\mathbf{v}) = k^{-1} \cdot T(\mathbf{v})$ is a similarity transformation with ratio of similitude $k^{-1} \cdot k - 1$ and hence S is an isometry of $\mathbb{R}^n$. Therefore, by Theorem II.4.9 and Remark 4 on page 71 of the notes, we know that S preserves angle measures. If we combine this with item (2), we find that $T = k \cdot S$ also preserves angle measures.$\blacksquare$

Finally, we derive (4). If $k > 0$ and T_k is defined as in (1), let $D = T_k(A)$, $E = T_k(B)$, and $F = T_k(C)$. Then by (1) and (2) we know that the affine transformation T_k maps ΔABC to a triangle ΔDEF such that $\Delta ABC \sim \Delta DEF$ with ratio of similitude k.$\blacksquare$