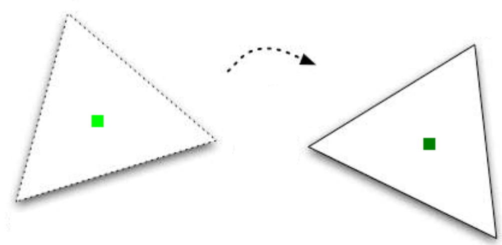
Calculus and the centroid of a triangle

The goal of this document is to verify the following physical assertion, which is presented immediately after the proof of Theorem **III.4.1**:

> *If* **X** *is the solid triangular region in* $\mathbb{R}^2$ *of uniform density whose vertices are the noncollinear points* **A, B** *and* **C,** *then the center of mass for* **X** *is given by* $(^1/_3)\cdot$**[A + B + C].**

There are two parts to our argument, the first of which is the following "physically obvious" statement about the effect of rigid motions (or Galilean transformations) on centroids:

<u>Theorem 1.</u> *Let* **X** *be a subset of* $\mathbb{R}^2$ *which is measurable in the sense that one can define its area by some reasonable method, let* **z** *denote the centroid of* **X** *defined by the usual sorts of formulas from integral calculus, and let* **G** *be a Galilean transformation of* $\mathbb{R}^2$*. Then* **G(z)** *is the centroid of the subset* **G[X].**

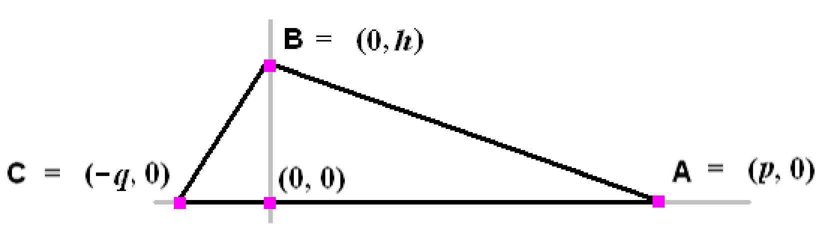


(Source: http://www.math.cornell.edu/~mec/Summer2008/youssef/Groups/images/triangle_rotation.jpg)

Although this statement corresponds to everyday experience with physical objects, writing up a full mathematical proof is considerably more complicated than one might expect (among other things, it is necessary to be precise about a reasonable method for defining area). Details will be given at the end of this document.

If **G** is a Galilean transformation of $\mathbb{R}^2$, then the results of Unit **II** show that **G** sends $(^1/_3)\cdot$**[A + B + C]** to $(^1/_3)\cdot$**[G(A) + G(B) + G(C)],** and therefore the if the centroid formula is true for **X** then it is also true for **G[X].** Consequently, it will suffice to prove the centroid formula for a class **K** of triangles △**DEF** such that every triangle is congruent to a triangle in F. The second part of the verification is to show that the formula is true for a suitable family of this type.

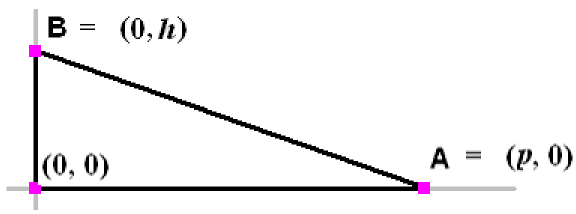
<u>CLAIM.</u> *Every triangle* △**ABC** *in* $\mathbb{R}^2$ *is congruent to a triangle whose vertices are* $(p, 0)$*,* $(0, h)$ *and* $(-q, 0)$*, where* p *and* h *are positive real numbers and* q *is a nonnegative real number.*



Proof of the claim. We know that the triangle has at least two acute angles, and without loss of generality we may as well assume the vertices of two such angles are **A** and **C.** In this case Corollary **III.4.1** implies that the foot **D** of the perpendicular from **B** to **AC** lies between **A** and **C** (see the drawing above). Then, as the drawing suggests, there is a Galilean transformation **G** be a Galilean transformation sending line **AC** to the $x-$axis such that **D** is sent to the origin, the first coordinate of **A** is negative, the first coordinate of **C** is positive, and the second coordinate of **D** is positive (the explicit construction of **G** is left to the reader as an exercise).■

By the preceding discussion, the verification of the centroid formula reduces to doing so for all triangles like those in the drawing; in other words, we need to show that the standard integral calculus formulas yield the value $(^1/_3)\cdot(p-q,h)$ for the centroid of the closed triangular region bounded by △**ABC.**

For the sake of simplifying the algebra, we shall first consider the special case where $q = 0$.



By construction the line **AB** is defined by the equation $y = h - (hx/p)$, and the standard centroid formulas from integral calculus immediately yield the moment of the solid triangular region with respect to the $y-$axis. Similarly, if we rewrite the equation for the line in the form $x = p - (py/h)$, we get the moment of the triangular region with respect to the $x-$axis:

<u>Formula 2.</u> The *moments of the solid triangular region with respect to the* $x-$ *and* $y-$ *axes are equal to* $(^1/_6)\cdot p^2h$ *and* $(^1/_6)\cdot ph^2$ *respectively.*

The derivation of this formula is a routine exercise in integral calculus.■

<u>Derivation of the centroid formula for the triangles in the Claim.</u> Let **M(x)** and **M(y)** denote the moments for the solid triangular region of △**ABC** with respect to the $y-$ and $x-$axes, let $\mathbf{M_+(x)}$ and $\mathbf{M_+(y)}$ denote the corresponding moments for the solid triangular region of △**ABD,** and let $\mathbf{M_-(x)}$ and $\mathbf{M_-(y)}$ denote the corresponding moments for the solid triangular region of △**ACD.** The moments $\mathbf{M_+(x)}$ and $\mathbf{M_+(y)}$ are given by Formula 2, and a similar argument shows that $\mathbf{M_-(x)}$ and $\mathbf{M_-(y)}$ are given by $-(^1/_6)\cdot q^2h$ and $(^1/_6)\cdot qh^2$ respectively. Therefore the total moments are given as follows:

$$\mathbf{M(x)} = \mathbf{M_+(x)} + \mathbf{M_-(x)} = (^1/_6)\cdot p^2h - (^1/_6)\cdot q^2h$$
$$\mathbf{M(y)} = \mathbf{M_+(y)} + \mathbf{M_-(y)} = (^1/_6)\cdot ph^2 + (^1/_6)\cdot qh^2$$

The area of the solid triangular region for △**ABC** is given by $\mathbf{S} = ½(p+q)h$, and therefore the coordinates of the centroid are given by $\mathbf{x^*} = \mathbf{M(x)/S} = (p+q)/3$ and $\mathbf{y^*} = \mathbf{M(y)/S} = h/3$, which is what we wanted to prove.■

Centroids and Galilean transformations of the plane

The preceding results show that the proof the centroid formula for an arbitrary solid triangular region in the plane will follow once we have verified Theorem 1. We begin by noting that if the conclusion of the theorem is valid for two Galilean transformations $\mathbf{G_1}$ and $\mathbf{G_2}$, then it follows immediately that the conclusion is also true for the composite transformation $\mathbf{G_1 G_2}$. Since an arbitrary Galilean transformation of the plane can be expressed as a composite of

(1) a ***translation*** $\mathbf{G(x)} = \mathbf{x} + \mathbf{v_0}$, where $\mathbf{v_0}$ is some vector in the plane,

(2) the ***reflection*** $\mathbf{G}(x, y) = (x, -y)$,

(3) a ***rotation*** through some angle θ,

this means it will suffice to prove the theorem for transformations of these three types, and we shall verify each case individually.

Rigid motions and moments for point masses

If $\mathbf{W}$ is a closed region of uniform density in the plane, then the $x-$ and $y-$ coordinates of its centroid are given by first computing its moment integrals with respect to these axes and then dividing each by the area of $\mathbf{W}$. The computations of the moment integrals follow a standard approach for applying integrals to measure quantities. Specifically, one starts by cutting $\mathbf{W}$ into a large number of small nonoverlapping pieces, so that the total moments are the sums of the moments of the pieces. Next, one approximates the moments of the pieces by moments of small point masses, where the points in question belong to the respective pieces and the mass of a point is just the area of that piece. Finally, one takes limits over such decompositions of $\mathbf{W}$ as the maximum diameter of the pieces goes to zero. Fundamental results from the theory of real variables, combined with reasonable assumptions about the values of moments, then imply that the limits give integral formulas for the moments of $\mathbf{W}$ with respect to the coordinate axes.

The remarks in the preceding paragraph suggest that a crucial step in verifying Theorem 1 is to compute the effect of a Galilean transformation on the moments of a point mass.

<u>**Theorem 3.**</u> *Let* $\mathbf{P} = (x, y)$ *be a point in* $\mathbb{R}^2$ *with mass equal to* m, *let* $\mathbf{M_1(P)}$ *and* $\mathbf{M_2(P)}$ *denote its moments with respect to the* $y-$ *and* $x-$ *axes, and let* $\mathbf{G}$ *be a Galilean transformation of* $\mathbb{R}^2$ *which is a translation, reflection or rotation as above. Then the moments of* $\mathbf{Q} = \mathbf{G(P)}$ *with respect to the coordinate axes are given as follows:*

(1) *If* $\mathbf{G(x)} = \mathbf{x} + \mathbf{v_0}$, *where* $\mathbf{v_0} = (a, b)$, *then* $\mathbf{M_1(Q)} = \mathbf{M_1(P)} + am$ *and* $\mathbf{M_2(Q)} = \mathbf{M_2(P)} + bm$.

(2) *If* $\mathbf{G}(x, y) = (x, -y)$, *then* $\mathbf{M_1(Q)} = \mathbf{M_1(P)}$ *and* $\mathbf{M_2(Q)} = -\mathbf{M_2(P)}$.

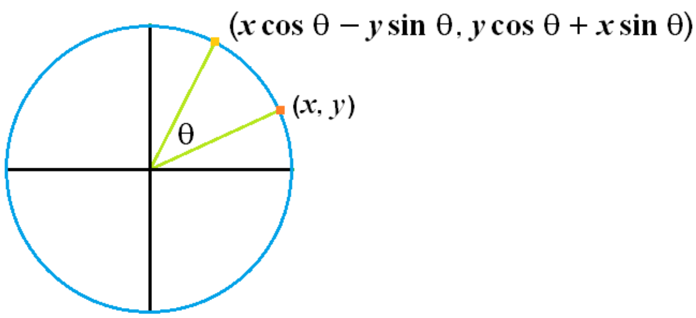
(3) *If* $\mathbf{G}$ *is a counterclockwise rotation through some angle* θ, *then we have* $\mathbf{M_1(Q)} = \mathbf{M_1(P)}\cos\theta - \mathbf{M_2(P)}\sin\theta$ *and* $\mathbf{M_2(Q)} = \mathbf{M_2(P)}\cos\theta + \mathbf{M_1(P)}\sin\theta$.

Proof. We shall verify the assertions in the stated order.

Verification of **(1).** By definition $\mathbf{M_1(P)} = xm$ and $\mathbf{M_2(P)} = ym$, and for the same reasons $\mathbf{M_1(Q)} = (x+a)m$ and $\mathbf{M_2(Q)} = (y+b)m$.

Verification of **(2).** Once again $\mathbf{M_1(P)} = xm$ and $\mathbf{M_2(P)} = ym$, and for the same reasons $\mathbf{M_1(Q)} = xm$ and $\mathbf{M_2(Q)} = -ym$.

Verification of **(3).** This case is slightly less trivial. As the drawing below suggests, a rotation through an angle θ sends $\mathbf{P} = (x, y)$ to $\mathbf{Q} = (x\cos\theta - y\sin\theta\,,\ y\cos\theta + x\sin\theta)$.



By definition the moments at **Q** are equal to m times its coordinates, and this yields the formula in the remaining case.■

Proof of Theorem 1

Since Galilean transformations preserve areas, we know that the areas of **X** and **G[X]** are equal to the same value which we shall call **C.** The next step is to compute the moments of **G[X]** with respect to the standard coordinate axes, and if we apply Theorem 3 we see that these moments given by the integrals on the next page. In analogy with previous notation, if **W** is a closed region of uniform density in the plane, then $\mathbf{M_1(W)}$ and $\mathbf{M_2(W)}$ will are its moments with respect to the $y-$ and $x-$ axes.

The integral formulas defining the moments of W with respect to the $y-$ and $x-$axes are given by

$$M_1(W) \;=\; \int\!\!\int_W u\,du\,dv\;, \qquad M_2(W) \;=\; \int\!\!\int_W v\,du\,dv$$

and the centroid coordinates are given by

$$(\overline{u},\overline{v}) \;=\; \left(\frac{M_1(W)}{C},\frac{M_2(W)}{C}\right)\;.$$

Similarly, the moments of $G[W]$ with respect to these axes are given as follows in the three basic cases. Throughout this discussion we adopt the notational convention $G(u,v) = (x,y)$.

Translations. For translations we have $x = u + a$ and $y = v + b$, so that

$$M_1(G[W]) \;=\; \int\!\!\int_W u + a\,du\,dv \;=\; M_1(W) \;+\; aC\;,$$

$$M_2(G[W]) \;=\; \int\!\!\int_W v + b\,du\,dv \;=\; M_2(W) \;+\; bC\;.$$

If we divide both of these equations by the area C, we see that the centroid of $G[W]$ has coordinates $(\overline{u} + a, \overline{v} + b)$.

Reflections. For the reflection in the second case we have $x = u$ and $y = -v$, so that

$$M_1(G[W]) \;=\; \int\!\!\int_W u\,du\,dv \;=\; M_1(W)\;,$$

$$M_2(G[W]) \;=\; \int\!\!\int_W -v\,du\,dv \;=\; -M_2(W)\;.$$

If we divide both of these equations by the area C, we see that the centroid of $G[W]$ has coordinates $(\overline{u}, -\overline{v})$.

Rotations. For the rotations in the second case we have $x = u\cos\theta - v\sin\theta$ and $y = v\cos\theta + u\sin\theta$, so that

$$M_1(G[W]) \;=\; \int\!\!\int_W u\cos\theta - v\sin\theta\,du\,dv \;=\; M_1(W)\cos\theta \;-\; M_2(W)\sin\theta\;,$$

$$M_2(G[W]) \;=\; \int\!\!\int_W v\cos\theta + u\sin\theta\,du\,dv \;=\; M_2(W)\cos\theta \;+\; M_1(W)\sin\theta\;.$$

If we divide both of these equations by the area C, we see that the centroid of $G[W]$ has coordinates

$$(\overline{u}\cos\theta - \overline{v}\sin\theta, \overline{v}\cos\theta + \overline{u}\sin\theta) \;=\; G(\overline{u},\overline{v})\;.\blacksquare$$