SOLUTIONS TO ADDITIONAL EXERCISES FOR II.1 AND II.2

Here are the solutions to the additional exercises in `betsepexercises.pdf`.

B1. Let $\mathbf{y}$ and $\mathbf{z}$ be distinct points of L; we claim that $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are not collinear. If there were some line M containing them, then we would have $M = L$ since both lines contain the last two points; however, we know that $\mathbf{x} \notin L$, so this is impossible.

To show the existence of a plane containing L and $\mathbf{x}$, let P be the unique plane containing $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$. Since $\mathbf{y}$ and $\mathbf{z}$ are in P, the axioms imply that the line joining them, which is L, must be contained in P. To see that there is only one plane containing L and $\mathbf{x}$, notice that a plane Q which contains both of these will automatically contain $\mathbf{y}$ and $\mathbf{z}$. Since there is only one plane containing $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, it follows that Q must be identical to P.■

B2. By the previous exercises, there is a unique plane P containing X and K. Let A, B, C be the points where K meets the lines L, M, N respectively. Then we have the following:

(1) Since X and A lie on P, the line $XA = L$ is contained in P.

(2) Since X and B lie on P, the line $XB = M$ is contained in P.

(3) Since X and C lie on P, the line $XC = N$ is contained in P.

Therefore P contains each of the lines K, L, M, N.■

B3. Since $A * B * C$ and $X * Y * Z$ are assumed, the conditions on the distances imply that

$$d(A, B) \;=\; d(A, C) \;-\; d(B, C) \;=\; d(X, Z) \;-\; d(Y, Z) \;=\; d(X, Y)$$

which is what we wanted to prove.■

B4. If $X \in (AB)$, then $A * X * B$ is true. By assumption, we have $A * B * C$ and therefore Proposition II.4 implies that $A * X * C$ is true, so that $X \in (AC)$. [*Note:* We are actually using an alternate form of this result; namely, $W * U * T$ and $W * V * U$ imply $W * V * T$. However, this follows from the stated form — $T * U * W$ and $U * V * W$ imply $T * V * W$ because $P * Q * R$ and $R * Q * P$ are equivalent conditions.]

B5. Both of the rays $[AB$ and $[BA$ are contained in the line AB, so we have $[AB \cup [BA \subset AB$. Conversely, suppose that $X \in AB$, and write $X = A + t(B - A)$ for some scalar t. If $t \neq 0$ then $X \in [AB$, while if $t < 0$ then we have $X * B * A$, and in fact we also have

$$X \;=\; B \;+\; (1 - t)(B - A)\;.$$

Since $t < 0$, it follows that $1 - t > 1$ and therefore $X \in [BA$.■

B6. Since $A * B * C$ holds, we know that $C = A + v(B - A)$ where $v > 1$.

If $X \in [AB]$, then $X = A + t(B - A)$ where $0 \leq t \leq 1$ and hence $X \in [AB$. If $X \in [BC$, then $X = B + s(C - B)$ where $s \geq 0$; using the equation in the preceding paragraph, we may use this to rewrite X as a linear combination of A and B as follows:

$$X \;=\; B + s[A + v(B - A) - B] \;=\; (1 + vs - s)B + (s - vs)A \;=\; A + (1 + vs - s)(B - A)$$

Since $s \geq 0$ and $v > 1$, it follows that $1 + s - vs > 1$, and therefore we see that $X \in [AB$. Hence $[AB] \cup [BC$ is contained in $[AB$.

Conversely, suppose that $X \in [AB$ and write $X = A + t(B - A)$ where $t \geq 0$. If $t \leq 1$, then we know that $X \in [AB]$. Suppose now that $t > 1$. By the equation in the first paragraph we have

$$A \;=\; \frac{1}{1 - v}\, C \;+\; \frac{-v}{1 - v}\, B$$

and therefore after substitution and some algebraic calculation we may rewrite X as a linear combination of B and C as follows:

$$X \;=\; B \;+\; \frac{1 - t}{1 - v}\,(C - B)$$

Since $t, v > 1$ it follows that the numerator and denominator of $(1 - t)/(1 - v)$ are both negative, so that the quotient is positive, and therefore it follows that X must lie on $[BC$. Hence we have $[AB \subset [AB] \cup [BC$, and if we combine this with the previous paragraph we conclude that $[AB = [AB] \cup [BC$.■

B7. We shall follow the hint and eliminate all of the alternatives. In both cases the points X and Y are on M but not equal to A, and since L and M can only have the point A in common it follows that neither X nor Y lies on L. Therefore in each case either X and Y lie on the same side of L or else they lie on opposite sides of L.

For part (a), we are given that $A * X * Y$, and we want to show that X and Y cannot lie on opposite sides of L. However, if they did, then there would be some point C such that $C \in L$ and $X * C * Y$. Now C would have to be a point of M, and since A is the only common point of L and M it would follow that $A = C$, so that $X * A * Y$. However, we know that $A * X * Y$, and thus we cannot have $X * A * Y$. This is a contradiction, and the source is our assumption that X and Y were on opposite sides of L; hence they must be on the same side of L.■

For part (b), we are given that $X * A * Y$, and we want to show that X and Y cannot lie on the same side of L. But if they did, then by convexity all points of (XY) would also lie on that half-plane, and we know that $A \in (XY) \cap L$ does not. This is a contradiction, and the source is our assumption that X and Y were on the same side of L; hence they must be on opposite sides of L.■

B8. We first observe that all points of $(AC$ lie on a common side of AB, and likewise for $(BD$. If $X \in (AC$, then either $X = C$, $A * C * X$ or $A * X * C$ holds. In each case X lies on the same side of AB as C. The proof for $(BD$ can be obtained by replacing A and C with B and D respectively.

By assumption, C lies on one side of AB, say H, and D lies on the other, say K. We can now use the preceding paragraph to conclude that $(AC \subset H$ and $(BD \subset K$. Since H and K have no points in common, the same is true for $(AC$ and $(BD$. Furthermore, since AC meets AB in A and BD meets AB in B, it follows that A cannot lie on $[BD$ and B cannot lie on $[AC$. If we combine the conclusions of the preceding two sentences, we see that $[AC$ and $[BD$ have no points in common.■

B9. We know that $[AB]$ is contained in $\Delta ABC \cap AB$. We shall follow the hint and show that if $X \in \Delta ABC$ but $X \notin [AB]$, then $X \notin AB$.

If $X = C$, then the conclusion follows because $C \notin AB$ by assumption. We are now left with the cases where $X \in (AC)$ or $X \in (BC)$; since the argument in the second case is the same as the argument in the first with A replaced by B, it is enough to show that $X \notin AB$ if $X \in (AC)$. If we did have $X \in (AC)$ and $X \in AB$, then it would follow that the line L containing A and X would be equal to AB. But A, X, C all lie on a single lie by assumption, and this line must be $L = AB$, which means that all three vertices of ΔABC would lie on L. This is a contradiction, and the source is our assumption that (AC) and AB have a point in common. Therefore (AC) and $[AB]$ do not have any points in common; as noted before the same conclusion will follow for (BC) and $[AB]$, and thus we see that no points of $[AC] - \{A\}$ or $[BC] - \{B\}$ can lie on the line AB, so that $\Delta ABC \cap AB$ must be equal to $[AB]$.■

B10. In each of these problems, we need to rewrite the line equation in the form $g(x, y) = 0$, and then we need to compare the signs of $g(X)$ and $g(Y)$.

(a) In this case we may take $g(x, y) = 9x - 4y - 7$. We have $g(3, 6) = -3 < 0$ and $g(1, 7) = -26 < 0$, so the two points lie on the same side of L.■

(b) In this case we may take $g(x, y) = 3x - y - 7$. We have $g(8, 5) = 12 > 0$ and $g(-2, 4) = -29 < 0$, so the two points lie on opposite sides of L.■

(c) In this case we may take $g(x, y) = 2x + 3y + 5$. We have $g(7, -6) = 1 > 0$ and $g(4, -8) = -11 < 0$, so the two points lie on opposite sides of L.■

(d) In this case we may take $g(x, y) = 7x + 3y - 2$. We have $g(0, 1) = 1 > 0$ and $g(-2, 6) = 2 > 0$, so the two points lie on the same side of L.■