Readings for Unit V from Greenberg
(Introduction to non – Euclidean geometry)

V.1 : Facts from spherical geometry

Comments. There is a discussion closely related to spherical geometry on pages 544 – 547 of Greenberg, but it is done in a setting like that of Section **V.6** and as such uses concepts not yet introduced in these notes (see also the comments regarding Section **V.6**); the relationship between the geometry discussed in the cited passage from Greenberg and spherical geometry is described fairly explicitly on page 126 of Greenberg.

V.2 : Attempts to prove Euclid's Fifth Postulate

Supplementary background readings.

Greenberg: pp. 161 – 176

Comments. Chapters **4 – 6** of Greenberg correspond to this section and the next one in these notes. In particular, pages 161 – 176 give proofs of several basic geometric theorems without using the Parallel Postulate or vector methods.

V.3 : Neutral geometry

Supplementary background readings.

Greenberg: pp. 176 – 191, 209 – 228, 269 – 270

Comments. The first passage covers the definitions and basic properties of Saccheri quadrilaterals, Lambert quadrilaterals, and rectangles in neutral geometry, culminating in the proof of the All or Nothing Theorem. The second passage gives a more detailed account of the historical developments leading to the discovery of non – Euclidean geoemetry; additional historical background on the concepts of Saccheri and Lambert quadrilaterals appears on page 177. This chapter contains considerably more information than the course notes. The final passage includes some more advanced exercises.

V.4 : Angle defects and related phenomena

Supplementary background readings.

Greenberg : pp. 239 – 247, 249 – 257, 257 – 267, 270 – 276, 477 – 481

Comments. The first passage discusses the historical background for the discovery of non – Euclidean geometry, and the second covers the basic results on angle sums, angle defects, the hyperbolic **AAA** Congruence Theorem, and common perpendiculars for certain pairs of parallel lines. There is a discussion of asymptotic parallels in third passage, ending with the statement of a hyperbolic version of the Euclidean theorem on circumcenters (see Section **III.4** of the notes). The fourth passage contains exercises in Greenberg which are related to the material in this section and the next one; some are quite challenging. Finally, the fifth passage is a discussion of angle defects, hyperbolic area, and the angle of parallelism from a slightly more sophisticated viewpoint.

V.5 : Further topics in hyperbolic geometry

Supplementary background readings.

Greenberg : pp. 276 – 280, 289 – 313, 481 – 483, 487 – 537, 571 – 596

Comments.

Greenberg's coverage of hyperbolic geometry is far more detailed than the course notes. The first passage contains some challenging exercises related to this section and the previous one, and the last one discusses alternate sets of axioms. The ultimate confirmation that the Euclidean Parallel Postulate cannot be proved from the other axioms is discussed on pages 289 – 301, which also discusses the mathematical model for hyperbolic geometry due to E. Beltrami and F. Klein. Additional features of this model are presented on pages 308 – 311. Pages 481 – 483 describe two important types of curves in hyperbolic geometry (***horocycles*** and ***equidistant curves***) which are nearly as important as lines and circles but have not counterparts in Euclidean geometry. Finally, the material on pages 487 – 528 contains additional material on hyperbolic geometry, including the development of hyperbolic trigonometry (pp. 487 – 480), hyperbolic (non – Cartesian) coordinate systems (pp. 507 – 515), the theory of elementary constructions (pp. 520 – 528), and advanced exercises (pp. 528 – 537). Two other standard models of the hyperbolic plane (the so – called **Poincaré models**) are described on pages 301 – 308, and yet another (the so – called **Minkowski model**) is described on pages 311 – 313. References for further material on these models are given in the discussions for Sections **V.6** and **V.7.**

This also seems like a good place to note that the standard concept of logical independence for an axiom (***i.e.***, there is a model for the rest of the axioms in which the given statement is not valid) is due to Saccheri; this is noted in the footnote at the bottom of pages 218 – 219 in Greenberg.

V.6 : Subsequent developments

Supplementary background readings.

Greenberg : pp. 248 – 249, 371 – 374, 408 – 448, 483 – 487, 541 – 570, 571 – 596

Comments.

The first passage and pages 541 – 547 summarize the historical background surrounding the development of elliptic geometry. Pages 483 – 487 interpret some features of hyperbolic geometry in terms of the differential geometry of curves and surfaces, and the passage on pages 548 – 570 cover a wide range of further topics in differential geometry. Issues concerning non – Euclidean geometry and the geometry of the physical universe are discussed in pages 371 – 374 (there is a lengthy subsequent discussion on pages 374 – 382 involving the philosophy of mathematics, but the latter topic is outside the scope of the course). The passage on pages 571 – 596 deals with subsequent work about some abstract issues concerning the role of the real number system in our axiomatic development(s) of Euclidean and non – Euclidean geometry; it is difficult to describe this work in a clear and simple manner without introducing some concepts from undergraduate algebra, and the file

http://math.ucr.edu/~res/math133/nonmetric_setting.pdf

is an attempt to provide some additional perspective. Finally, pages 408 – 448 cover the algebraic approach to studying the geometric symmetries of Euclidean and non – Euclidean spaces.

V.7 : Non – Euclidean geometry in modern mathematics

Supplementary background readings.

Greenberg : pp. 313 – 333, 333 – 346, 483 – 487, 571 – 596

Comments.

The passage on pages 313 – 333 discusses several aspects of the material in the Poincaré model, and pages 333 – 346 discuss some relations between the Beltrami – Klein model and the topics in Unit **IV** of the notes. Exercises involving the models are given on pages 346 – 365. Pages 382 – 389 discuss the ways in which hyperbolic geometry relates to other branches of mathematics; several of the topics covered in this section of the notes are also covered there, but each account includes examples not in the other. Finally, several passages cited in the previous section are also highly relevant to this one.