II.3 : Measurement axioms

A large — probably dominant — part of elementary Euclidean geometry deals with questions about linear and angular measurements, so it is clear that we must discuss the basic properties of these concepts.

Linear measurement

In elementary geometry courses, linear measurement is often viewed using the lengths of line segments. We shall view it somewhat differently as giving the (shortest) distance between two points. Not surprisingly, in $\mathbf{R}^2$ and $\mathbf{R}^3$ we define this distance by the usual Pythagorean formula already given in Section **I.1.** In the synthetic approach, one may view distance as a primitive concept given formally by a function d; given two points **X, Y** in the plane or space, the distance d(**X, Y**) is assumed to be a nonnegative real number, and we further assume that distances have the following standard properties:

<u>Axiom D – 1 :</u> *The distance* d(**X, Y**) *is equal to zero if and only if* **X = Y.**

<u>Axiom D – 2 :</u> *For all* **X** *and* **Y** *we have* d(**X, Y**) = d(**Y, X**).

Obviously it is also necessary to have some relationships between an abstract notion of distance and the other undefined concepts introduced thus far; namely, lines and planes. Intuitively we expect a geometrical line to behave just like the real number line, and the following axiom formulated by G. D. Birkhoff (1884 – 1944) makes this idea precise:

<u>Axiom D – 3</u> <u>(*Ruler Postulate*)</u> **<u>:</u>** *If* **L** *is an arbitrary line, then there is a* $1-1$ *correspondence between the points of* **L** *and the real numbers* $\mathbf{R}$ *such that if the points* **X** *and* **Y** *on* **L** *correspond to the real numbers* x *and* y *respectively, then we have*

$$d(\mathbf{X}, \mathbf{Y}) = |x - y|.$$

It is not difficult to define such a $1-1$ correspondence explicitly in the analytic approach. Suppose we are given a line containing the two points **A** and **B.** Then every point **X** on the line can be expressed uniquely as a sum **A** + k(**B – A**) for some real number k. The desired mapping is the one sending the point **X** to the number $x = k\,d(\mathbf{A}, \mathbf{B})$. A proof that this defines a $1-1$ correspondence is given in the exercises.

For the time being, we shall only give one simple but often needed result about linear measurement.

<u>Proposition 1.</u> *Let* **A** *and* **B** *be distinct points, and let* x *be a positive real number. Then there is a unique point* **Y** *on the open ray* **(AB** *such that* $d(\mathbf{A}, \mathbf{Y}) = x$. *Furthermore, we have* **A*Y*B** *if and only if* $x < d(\mathbf{A}, \mathbf{B})$, *and similarly we have* **A*B*Y** *if and only if* $x > d(\mathbf{A}, \mathbf{B})$.

Proof. We begin by proving the existence and uniqueness assertions in the first part of the conclusion. Let $c = d$**(A, B),** so that c and c^{-1} are both positive. If we take

$$\mathbf{Y} = \mathbf{A} + xc^{-1}(\mathbf{B} - \mathbf{A})$$

then **Y** lies on **(AB** and straightforward computation shows that

$$d(\mathbf{A}, \mathbf{Y}) = ||\mathbf{Y} - \mathbf{A}|| = ||xc^{-1}(\mathbf{B} - \mathbf{A})|| = xc^{-1}||(\mathbf{B} - \mathbf{A})|| = x.$$

Furthermore, if $\mathbf{Z} = \mathbf{A} + w(\mathbf{B} - \mathbf{A})$ is an arbitrary point on **(AB** such that $d(\mathbf{A}, \mathbf{Z}) = x$, then we have $d(\mathbf{A}, \mathbf{Z}) = w\,d(\mathbf{A}, \mathbf{B})$, which implies that $w = xc^{-1}$ and hence $\mathbf{Z} = \mathbf{Y}$.

We shall next consider the "if" implication in the second part of the proposition. The conditions **A*Y*B** and **A*B*Y** imply $x < d(\mathbf{A}, \mathbf{B})$ and $x > d(\mathbf{A}, \mathbf{B})$ respectively, so this part is easy. To prove the converse, note that every point **Y** on **(AB** satisfies exactly one of the three conditions **A*Y*B, Y = B,** or **A*B*Y.** Thus $x < d(\mathbf{A}, \mathbf{B})$ can only happen if we have **A*Y*B,** and likewise $x > d(\mathbf{A}, \mathbf{B})$ can only happen if **A*B*Y.**■

Angles and their interiors

We have not yet defined angles, but we shall do so now.

Definition. Let **(A, B, C)** be an ***ordered*** triple of noncollinear points. The ***angle*** **ABC,** generally written **∠ABC,** is equal to the union of the rays **[BA** and **[BC.**

This definition **excludes** the extreme concepts of a ***zero – degree angle*** for which the two rays are equal and also a ***straight angle*** in which the two rays are opposite rays on the same line.

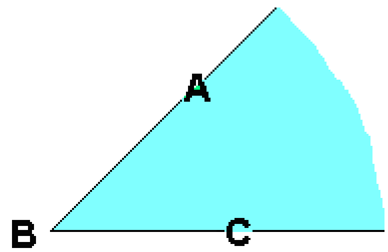
It follows immediately that **∠CBA = ∠ABC.** Note that ***this is stronger than saying the angles have the same measurements*** (which we have not yet discussed formally); it means that ***the two angles consist of exactly the same points.*** Here is another basic statement about identical angles:

Proposition 2. *If* **D** *lies on the open ray* **(BA** *and* **E** *lies on the open ray* **(BC,** *then we have* **∠DBE = ∠ABC.**

This follows because we have **[BA = [BD** and **[BC = [BE.**■

Definition. If **∠ABC** is an angle, then its ***interior,*** written **Int ∠ABC,** is equal to the intersection **H ∩ K,** where **H** is the half plane determined by **BA** which contains **C,** and **K** is the half plane determined by **BC** which contains **A.** Frequently one uses phrases such as, "**X** lies ***inside*** **∠ABC,**" to express the relationship **X ∈ Int ∠ABC.** It is also possible to define the ***exterior*** of the angle, written **Ext ∠ABC,** as all points in the plane not on the angle or in its interior; phrases like "**X** lies ***outside*** **∠ABC**" express the relationship **X ∈ Ext ∠ABC.**

In the figure below, the shaded region corresponds to the interior of **∠ABC.** Of course, the interior also extends indefinitely beyond the shaded part to the upper right.

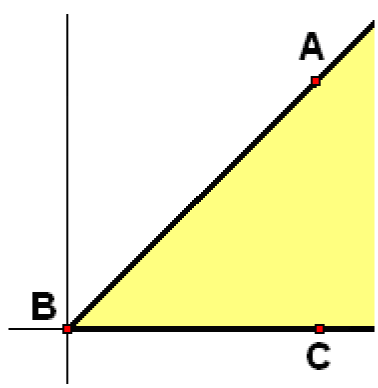


The following criterion for recognizing points in the interior of ∠**ABC** is often extremely useful**:**

Proposition 3. *Let* **A, B, C** *be noncollinear points in* $\mathbf{R}^2$, *let* **D** $\in$ $\mathbf{R}^2$, *and express* **D** *using barycentric coordinates as* x**A** + y**B** + z**C.** *Then* **D** *lies in the interior of* ∠**ABC** *if and only if both* x *and* z *are positive*.

Proof. By previous results, a point **D** lies on the same side of **BA** as **C** if and only if z is positive, and it lies on the same side of **BC** as **A** if and only if x is positive, so the conclusion follows directly from these considerations and the definition of an angle's interior.■

Examples. We shall consider a case where it is intuitively clear which points lie in the interior of the given angle and show how this corresponds to the numerical criteria in the previous proposition. Let **A** = $(1, 1)$, **B** = $(0, 0)$ and **C** = $(1, 0)$. Then we expect the points in the interior of ∠**ABC** to be those points (p, q) such that $0 < q < p$.

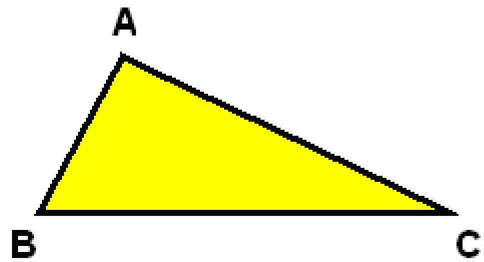


In this example **B** = **0** and thus the barycentric coordinates x and z for a point **D** are simply the coefficients for expressing a point as a linear combination of **A** and **C;** the third barycentric coordinate y is then equal to $1 - x - z$. But we have

$$(p, q) \quad = \quad q\,(1, 1) \;+\; (p - q)\,(1, 0)$$

and therefore the conditions in the proposition are that $q > 0$ and $p - q > 0$, which are equivalent to the previously stated inequalities.

We can use similar considerations to give a precise definition for the interior of a triangle.

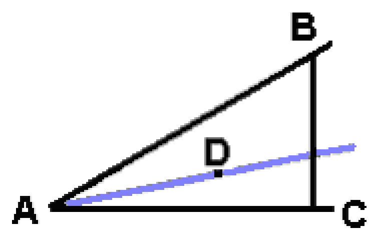


Formally, we define the ***interior of*** $\triangle$**ABC** similarly as the intersection of three half – planes $\mathbf{H_A} \cap \mathbf{H_B} \cap \mathbf{H_C}$, where $\mathbf{H_A}$ is the half – plane determined by **BC** which contains **A,** while $\mathbf{H_B}$ is the half – plane determined by **AC** which contains **B** and $\mathbf{H_C}$ is the half – plane determined by **AC** which contains **B.** By standard set – theoretic identities, this set is also equal to the intersection of the interiors of $\angle$**ABC,** $\angle$**BCA,** and $\angle$**CAB.** The following result is then an immediate consequence of the previous proposition:

<u>Corollary 4.</u> *Let* **A, B, C** *be noncollinear points in* $\mathbf{R}^2$, *let* $\mathbf{D} \in \mathbf{R}^2$, *and express* **D** *using barycentric coordinates as* $x\mathbf{A} + y\mathbf{B} + z\mathbf{C}$. *Then* **D** *lies in the interior of* $\triangle$**ABC** *if and only if all three barycentric coordinates* x, y *and* z *are positive.*■

The next result also arises frequently in elementary geometry but is not justified in classical treatments of the subject.

<u>Theorem 5.</u> <u>(Crossbar Theorem)</u> *Let* **A, B, C** *be noncollinear points in* $\mathbf{R}^2$, *and let* **D** *be a point in the interior of* $\angle$**CAB.** *Then the segment* **(BC)** *and the open ray* **(AD** *have a point in common.*



<u>Proof.</u> Using barycentric coordinates we may write $\mathbf{D} = x\mathbf{A} + y\mathbf{B} + z\mathbf{C}$, and the condition $\mathbf{D} \in \mathrm{Int}\,\angle\mathbf{CAB}$ implies that y and z are positive. The objective is to find a point **P** which is expressible both as $(1-t)\mathbf{B} + t\mathbf{C}$ for some t satisfying $0 < t < 1$ and as $(1-u)\mathbf{A} + u\mathbf{D}$ for some u satisfying $u > 0$. Let us see what happens if we set these expressions equal to each other. The formula for **D** implies a chain of equations having the form

$$(1-t)\mathbf{B} + t\mathbf{C} = (1-u)\mathbf{A} + u\mathbf{D} =$$

$$(1-u)\mathbf{A} + u(x\mathbf{A} + y\mathbf{B} + z\mathbf{C}) = (1-u+ux)\mathbf{A} + uy\mathbf{B} + uz\mathbf{C}$$

and if we equate coefficients we obtain $t = uy$ and $1 - t = uz$, which yield the following values for t and u:

$$u = \frac{1}{y+z} \quad , \quad t = \frac{z}{y+z}$$

Since y and z are positive, we have the inequalities $0 < z < y + z$, which in turn imply that $u > 0$ and $0 < t < 1$. If we reverse the preceding derivation, it will follow that $(1-t)\mathbf{B} + t\mathbf{C} = (1-u)\mathbf{A} + u\mathbf{D}$ for these choices of t and u.■

Here is one useful consequence of the Crossbar Theorem.

<u>**Proposition 6.**</u> <u>(Geometric Trichotomy Principle)</u> *Let* **A** *and* **B** *be distinct points in* $\mathbf{R}^2$, *and let* **C** *and* **D** *be two points on the same side of* **AB**. *Then exactly one of the following is true*:

(1) **D** *lies on* **(BC** (*equivalently, the open rays* **(BC** *and* **(BD** *are equal*).

(2) **D** *lies in* $\mathrm{Int}\,\angle$**ABC**.

(3) **C** *lies in* $\mathrm{Int}\,\angle$**ABD**.

<u>Proof.</u> As usual write $\mathbf{D} = x\mathbf{A} + y\mathbf{B} + z\mathbf{C}$ using barycentric coordinates. By the assumptions we know that z is positive. We claim that the three alternatives in the conclusion correspond to the mutually exclusive options $x > 0$, $x = 0$, and $x < 0$. If $x > 0$, the characterization of $\mathrm{Int}\,\angle$**ABC** in terms of barycentric coordinates shows that $\mathbf{D} \in \mathrm{Int}\,\angle$**ABC**. If $x = 0$, then the barycentric coordinate equation may be rewritten in the form $\mathbf{D} = \mathbf{B} + z(\mathbf{C} - \mathbf{D})$, which implies that $\mathbf{D} \in$ **(BC** because z is positive. Finally, suppose that $x < 0$; to prove the third alternative, we need to express **C** as a linear combination of **A, B** and **D** using barycentric coordinates and check that the coefficient of **A** is positive. One way of finding the barycentric expansion is to start by writing $\mathbf{C} - \mathbf{B}$ as a linear combination of $\mathbf{A} - \mathbf{B}$ and $\mathbf{D} - \mathbf{B}$ and to rearrange the terms afterwards. The formula for **D** implies

$$(\mathbf{D} - \mathbf{B}) = x(\mathbf{A} - \mathbf{B}) + z(\mathbf{C} - \mathbf{B})$$

and if we solve for **C – B** we obtain the equation

$$(\mathbf{C} - \mathbf{B}) = -xz^{-1}(\mathbf{A} - \mathbf{B}) + z^{-1}(\mathbf{D} - \mathbf{B})$$

which in turn implies the barycentric coordinate formula for **C:**

$$\mathbf{C} = z^{-1}\big(-x\mathbf{A} + \mathbf{B} + (x + z - 1)\mathbf{D}\big)$$

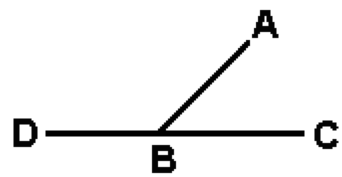
Since z is positive and x is negative by our assumptions, it follows that the barycentric coordinate of **A**, which equals $-xz^{-1}$, must be positive. As noted earlier, this means that $\mathbf{C} \in \mathrm{Int}\,\angle$**ABD**.■

Angular measurement

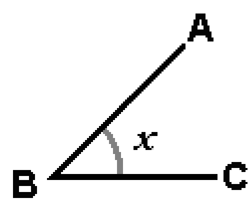
The basic axioms for angular measurement are considerably more complicated to state than any of the previous ones with the possible exceptions of the separation postulates. We have already given the analytic definition in Section **II.1.** In the synthetic approach, angle measurement is given formally as an abstract function μ, which assigns to each ordered triple of noncollinear points **(A, B, C)** a real number $\mu\angle$**ABC.** This value is always strictly between 0 and $180°$, and it called the ***angular measurement, angle measure,*** or something similar.

<u>**Axiom AM – 0** (*Invariance Property*) **:**</u> *If* $\angle$**DBE** *and* $\angle$**ABC** *are the same set, then* $\mu\angle$**DBE** = $\mu\angle$**ABC.**

<u>**Axiom AM – 1** (*Supplement Postulate*) **:**</u> *If* **D** *satisfies* **D∗B∗C** *then we have the identity* $\mu\angle$**ABD** + $\mu\angle$**ABC** = $180°$.

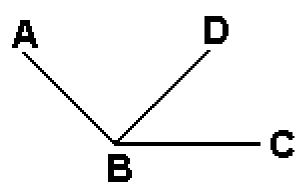


<u>**Axiom AM – 2** (*Protractor Postulate*) **:**</u> *If* $0 < x < 180°$ *and* **H** *is a half – plane associated to the line* **BC,** *then there is a unique ray* **[BA** *such that* **(BA** *is contained in* **H** *and* $\mu\angle$**ABC** = x.



<u>**Axiom AM – 3** (*Additivity Postulate*) **:**</u> *If* **D** *lies in* **Int** $\angle$**ABC,** *then we have*

$\mu\angle$**ABD** + $\mu\angle$**DBC** = $\mu\angle$**ABC.**

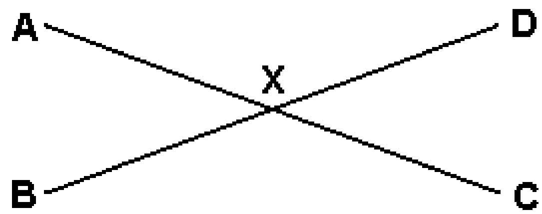


Frequently it is convenient to write $\mu\angle$**ABC** in the ***alternate form*** $|\angle$**ABC**$|$, and in fact the latter is the notation we shall generally use throughout these notes.

In a complete treatment of Euclidean geometry, it would be necessary to verify that the analytically defined angular measurement satisfies all these properties, but in our combined treatment we shall not do so (this would require very lengthy and complicated digressions). Details appear in the previously cited book by Moïse.

We shall only prove a few simple consequences of the angle measurement axioms here; they will be used extensively in the next unit. The next result may be one of the earliest in the systematic deductive formulation of geometry, and it is attributed to Thales of Miletus (*c.* 624 B.C.E. – *c.* 547 B.C.E.)**:**

<u>**Proposition 7.** (Vertical Angle Theorem)</u> *Let* **A, B, C, D** *be four distinct points such that* **A∗X∗C** *and* **B∗X∗D.** *Then* $\mu\angle$**AXB** = $\mu\angle$**CXD.**



Proof. Two applications of the Supplement Postulate imply that

$$\mu\angle \mathbf{AXB} + \mu\angle \mathbf{AXD} = 180° = \mu\angle \mathbf{DXA} + \mu\angle \mathbf{DXC}$$

and if we subtract $\mu\angle \mathbf{AXD} = \mu\angle \mathbf{DXA}$ from the left and right hand side we obtain

$$\mu\angle \mathbf{AXB} = \mu\angle \mathbf{DXC}$$

which is equivalent to the conclusion because $\angle \mathbf{CXD} = \angle \mathbf{DXC}$.■

The next result verifies an intuitively clear relationship between angle measurement and interiors of angles.

Theorem 8. *Let* **A, B, C, D** *be distinct coplanar points, and suppose that* **C** *and* **D** *lie on the same side of* **AB.** *Then* $\mu\angle \mathbf{CAB} < \mu\angle \mathbf{DAB}$ *if and only if* **C** *lies in the interior of* $\angle \mathbf{DAB}$.

Proof. The "if" direction is a consequence of the Additivity Postulate (which implies that $\mu\angle \mathbf{CAB} + \mu\angle \mathbf{DAC} = \mu\angle \mathbf{DAB}$) and the fact that $\mu\angle \mathbf{DAC}$ is positive. The reverse implication follows from the Additivity Postulate and the Trichotomy Principle. Specifically, if **C** does not lie in the interior of $\angle \mathbf{DAB}$, then either **C** lies on **(AD** or else **D** lies in the interior of $\angle \mathbf{CAB}$; in the first case we have $\mu\angle \mathbf{CAB} = \mu\angle \mathbf{DAB}$ and in the second we have $\mu\angle \mathbf{CAB} > \mu\angle \mathbf{DAB}$.■

II.4 : Congruence, superposition and isometries

Although the relationships between linear and angular measurement are a major theme in Euclidean geometry, the measurement axioms in the preceding section say nothing about any such relationships. In the synthetic approach it is necessary to have axioms which describe the ties between the two types of measurements. One of the fastest ways of doing so is to assume the three basic congruence principles for triangles as axioms. This is actually a bit redundant, for if one of them is true then the others can be derived from it. However, assuming all three will allow us to bypass a few logical detours. Later in this section we discuss some logical arguments involving congruence from the ***Elements*** and describe the underlying ideas from a modern perspective.

The triangle congruence principles

Not surprisingly, we start with two triangles $\triangle \mathbf{ABC}$ and $\triangle \mathbf{DEF}$. More accurately, we start two ordered triples of noncollinear points **(A, B, C)** and **(D, E, F)** where it is possible that the sets **{A, B, C}** and **{D, E, F}** may be identical (for example, we may have **D = B, E = C** and **F = A**).

<u>**Axiom SAS** (*Side – Angle – Side Postulate*)</u>**:** *Suppose we have two ordered triples* **(A, B, C)** *and* **(D, E, F)** *as above such that* d**(A, B)** = d**(D, E)**, d**(B, C)** = d**(E, F)**, *and* **|∠ABC|** = **|∠DEF|**. *Then we also have* d**(A, C)** = d**(D, F)**, **|∠BAC|** = **|∠EDF|**, *and* **|∠ACB|** = **|∠DFE|**.

In geometry one generally writes **△ABC ≅ △DEF** and says that **△ABC** and **△DEF** are ***congruent*** if the six equations in this postulate are satisfied. However, this is really a slight *<u>abuse of language</u>* because ***the orderings of the vertices are absolutely essential***; we may write **△ABC ≅ △DEF,** but in doing so we do not necessarily mean to assert that **△ABC ≅ △EDF** even though **△DEF** and **△EDF** are exactly the same triangle.

<u>**Axiom ASA** (*Angle – Side – Angle Postulate*)</u>**:** *Suppose we have ordered triples* **(A, B, C)** *and* **(D, E, F)** *as above satisfying the conditions* d**(B, C)** = d**(E, F)**, **|∠ABC|** = **|∠DEF|**, *and* **|∠ACB|** = **|∠DFE|**. *Then we have* **△ABC ≅ △DEF.**

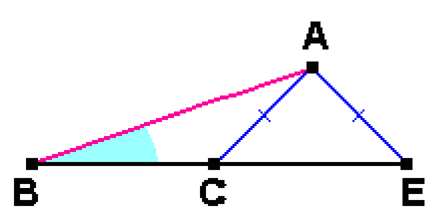
<u>**Axiom SSS** (*Side – Side – Side Postulate*)</u>**:** *Suppose we have ordered triples* **(A, B, C)** *and* **(D, E, F)** *as above such that* d**(A, B)** = d**(D, E)**, d**(B, C)** = d**(E, F)**, *and* d**(A, C)** = d**(D, F)**. *Then we have* **△ABC ≅ △DEF.**

At the beginning of this section we noted that ***if we assume just one of these three postulates, then we can prove the other two.*** For the sake of completeness, in the file

http://math.ucr.edu/~res/math133/trianglecongruence.pdf

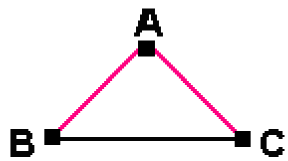
we shall give (synthetic) proofs that **SAS** implies both **ASA** and **SSS.**

There is also an **AAS** congruence principle (**△ABC ≅ △DEF** if d**(B, C)** = d**(E, F)**, **|∠ABC|** = **|∠DEF|**, and **|∠BAC|** = **|∠EDF|**) which can be deduced from the preceding statements, but we shall not do so here (one proof of this will be given in Section **III.2,** and another is given by an Exercise from Section **V.2**). On the other hand, *<u>there is</u>* **NO SSA** *<u>congruence principle</u>*; this is illustrated by the drawing below, in which we have d**(A, C)** = d**(A, E)**, so that **△CAB** and **△EAB** satisfy **SSA** but are not congruent.



The standard results involving isosceles triangles are immediate consequences of our axioms.

Theorem 1. (Isosceles Triangle Theorem) *In* $\triangle$**ABC**, *one has* d**(A, B)** = d**(A, C)** *if and only if* |∠**ABC**| = |∠**ACB**|.



Proof. If d**(A, B)** = d**(A, C)** then we can use the **SAS** assumption to conclude that $\triangle$**BAC** ≅ $\triangle$**CAB.** The latter implies that |∠**ABC**| = |∠**ACB**|. Conversely, if we have |∠**ABC**| = |∠**ACB**|, then we can use the **ASA** assumption to conclude that $\triangle$**ABC** ≅ $\triangle$**ACB,** and the latter implies that d**(A, B)** = d**(A, C)**.■

Corollary 2. *In* $\triangle$**ABC**, *one has* d**(A, B)** = d**(A, C)** = d**(B, C)** *(the triangle is* ***equilateral****) if and only if one has* |∠**ABC**| = |∠**ACB**| = |∠**BAC**| *(the triangle is* ***equiangluar****)*.

The corollary follows from two applications of the theorem.■

The idea of applying a congruence result to the same triangle with permuted vertices is essentially due to Pappus of Alexandria (*c.* 290 – *c.* 350). Euclid's proof of the "if" direction (Proposition **5** of Book **I**) is a fairly lengthy argument which requires the construction of auxiliary points and line segments, and it receives so much attention in books on geometry and the history of mathematics that a reference for it should be given; in particular, the proof is discussed on pages 151 – 152 of the following textbook:

> D. M. Burton, ***The history of mathematics: An introduction*** (Sixth Edition). *McGraw – Hill, Boston, MA,* 2005. ISBN: 0–07–305189–6.

One might speculate that Euclid did not realize that congruence theorems could be applied to the same triangle with permuted vertices, but there is no corroborating evidence for or against this (in fact, we know almost nothing about Euclid himself).

Congruence and the other axioms. In the ***Elements*** and quite a few other treatments of synthetic geometry, the basic congruence theorems are formulated as theorems rather than postulates. Two of the reasons for this difference will be mentioned here. Historically these proofs were based upon an intuitive idea of superposition that we shall discuss at some length after the end of this paragraph. In another direction, recall our earlier observation that the linear and angular measurement axioms say nothing about how these two types of measurement interact. The lack of such links should raise doubts whether one could hope to prove any (equivalently, all) of the basic congruence theorems for triangles. In fact, it is possible to prove rigorously that **SAS, ASA** and **SSS** cannot be logically derived from the remaining postulates that were stated in the preceding sections (and this also holds if we add the postulate that will be stated in Section **5**). We shall explain this statement at the end of this section in an Appendix.

Classical superposition

> Coincidence [superposition] is either mere tautology, or something entirely empirical, which belongs ... to external sensuous experience.
>
> A. Schopenhauer (1788 – 1860), ***The World as Will*** **(1818)**

The proofs of **SAS** and **SSS** in the ***Elements*** are exceptional because they rely on a principle of moving an object without changing its size or shape. In both cases one starts with △**ABC** and △**DEF** such that |∠**ABC**| = |∠**DEF**|, and the idea is to move so that side **[BA]** will lie on ray **[ED** and side **[BC]** will lie on ray **[EF;** by construction the vertex **B** is sent to **E.** The goal is then to show that the images of **A** and **C** are precisely the points **D** and **F,** so that we have a ***superposition*** of △**ABC** directly on top of △**DEF,** and from this to conclude that the corresponding parts of the triangles have equal measurements. The following online sites contain several interactive videos illustrating the physical concept of superposition:

http://www.ies.co.jp/math/products/geo1/menu.html (see the subheading, ***Congruent Figures and Triangles***)

http://standards.nctm.org/document/eexamples/chap6/6.4/index.htm

http://standards.nctm.org/document/eexamples/chap6/6.4/part3.htm

http://standards.nctm.org/document/eexamples/chap6/6.4/part4.htm

Everyday experience with physical objects strongly suggests such rigid motions of objects are easy to achieve. However, we are dealing here with ***mathematical objects*** rather than ***physical objects,*** so to be logically complete we must either deduce this somehow from our setting for geometry or else make an additional assumption to justify it. As noted before, there is speculation that Euclid may have been uncomfortable with the use of superposition, for there are numerous other places in the ***Elements*** where it could have been used equally well.

A modern approach

> Congruent parts of space **V, V′** are such as can be occupied by the same rigid body in two of its positions. If you move the body from one into the other position, the particle of the body covering point **P** of **V** will afterwards cover a certain point **P′** of **V′**, and thus the result of the motion is a mapping **P** → **P′** of **V** upon **V′**. We can extend the rigid body either actually or in imagination so as to cover an arbitrarily given point **P** of space, and hence the congruent mapping **P** → **P′** can be extended to the entire space.
>
> H. Weyl (1885 – 1955)

The first step in analyzing the notion of superposition is to find a mathematical model for the notion of moving an object. The concept of ***function*** is excellently suited for this purpose. Suppose that we have some geometric object, which we view as a subset **K** of $\mathbf{R}^n$, and we want to see what happens when we move it. A physical motion can be modeled mathematically by a function **f** which is defined on all points of **K** and takes values in $\mathbf{R}^n$. Most motions at this level of generality will not preserve any geometrical properties of **K**. If we want to preserve any geometrical properties, we need to make some assumptions on the function. For our purposes the following seem pretty basic:

1. The function **f** is $1-1$ (so two points do not get squashed into a single point).
2. If **x** and **y** are points of **K**, then **f** preserves the distance between them; in other words, we have $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y}))$.
3. The function **f** sends collinear points to collinear points and noncollinear points to noncollinear points.
4. If **x, y, z** are noncollinear points of **K**, then **f** preserves the measurement of the angle they form; in other words, we have $|\angle \mathbf{xyz}| = |\angle \mathbf{f(x)\,f(y)\,f(z)}|$.

We know that the ***inclusion mapping*** from **K** into $\mathbf{R}^n$ has these properties for trivial reasons; physically, this corresponds to the "motion" which does not move anything at all. Clearly we are more interested in having examples where the function actually changes something. Linear algebra provides powerful methods for attacking such questions.

Isometries and linear algebra

We shall be interested in the following class of mappings from $\mathbf{R}^n$ to itself:

Definition. A function (or mapping) **T** from $\mathbf{R}^n$ to itself is said to be an ***isometry*** if it is a $1-1$ onto map such that $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{T}(\mathbf{x}), \mathbf{T}(\mathbf{y}))$ for all vectors **x** and **y** in $\mathbf{R}^n$.

Before we investigate the relevance of such mappings to the issues raised above, we shall study them briefly as objects in their own right.

Proposition 3. *The identity map is an isometry from $\mathbf{R}^n$ to itself. If **T** is an isometry from $\mathbf{R}^n$ to itself, then so is its inverse $\mathbf{T}^{-1}$. Finally, if **T** and **U** are isometries from $\mathbf{R}^n$ to itself then so is their composite $\mathbf{T} \circ \mathbf{U}$.*

Proof. These arguments may be familiar, but we include them for the sake of completeness. The first part follows from the tautology $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, \mathbf{y})$. To prove the second assertion, let $\mathbf{u} = \mathbf{T}^{-1}(\mathbf{x})$ and $\mathbf{v} = \mathbf{T}^{-1}(\mathbf{y})$, so that $\mathbf{x} = \mathbf{T}(\mathbf{u})$ and $\mathbf{y} = \mathbf{T}(\mathbf{v})$. We then have

$$d(\mathbf{T}^{-1}(\mathbf{x}), \mathbf{T}^{-1}(\mathbf{y})) = d(\mathbf{u}, \mathbf{v}) = d(\mathbf{T}(\mathbf{u}), \mathbf{T}(\mathbf{v})) = d(\mathbf{x}, \mathbf{y})$$

which is what we need to verify. In the last case we apply the isometry condition for **T** and **U** in separate steps to conclude that

$$d(\mathbf{T} \circ \mathbf{U}\,(\mathbf{x}), \mathbf{T} \circ \mathbf{U}\,(\mathbf{y})) = d(\mathbf{U}(\mathbf{x}), \mathbf{U}(\mathbf{y})) = d(\mathbf{x}, \mathbf{y})$$

once again obtaining the desired conclusion.■

Examples of isometries. **1.** If **T** is a linear transformation from $\mathbf{R}^n$ to itself and is given by the square matrix **A** (so that $\mathbf{T(x)} = \mathbf{Ax}$), then **T** is a rigid motion from $\mathbf{R}^n$ to itself if **A** is an ***orthogonal*** matrix; in other words, the inverse matrix $\mathbf{A}^{-1}$ is equal to the transposed matrix ${}^{\mathbf{T}}\mathbf{A}$. This is true because

$$\langle \mathbf{Ax}, \mathbf{Ax} \rangle = {}^{\mathbf{T}}(\mathbf{Ax})\,\mathbf{Ax} = {}^{\mathbf{T}}\mathbf{x}\,{}^{\mathbf{T}}\mathbf{AAx} = {}^{\mathbf{T}}\mathbf{x}\,\mathbf{I}\,\mathbf{x} = \left({}^{\mathbf{T}}\mathbf{x}\right)\mathbf{x} = \langle \mathbf{x}, \mathbf{x} \rangle$$

shows that **T** preserves lengths of vectors, so that

$$d(\mathbf{T(x)}, \mathbf{T(y)}) = ||\mathbf{T(x)} - \mathbf{T(y)}|| = ||\mathbf{T(x - y)}|| = ||\mathbf{x - y}|| = d(\mathbf{x}, \mathbf{y}).$$

In fact, the converse is also true by results from linear algebra; if a linear transformation defines an isometry then it is given by an orthogonal matrix. Linear algebra courses provide many examples of orthogonal matrices; one characterization of such matrices is that their rows and/or columns form an orthonormal basis for $\mathbf{R}^n$.

2. If **w** is an arbitrary vector in $\mathbf{R}^n$ then the ***translation mapping*** $\mathbf{S_w}$ from $\mathbf{R}^n$ to itself defined by $\mathbf{S_w(x)} = \mathbf{x + w}$ is also an isometry. This is a consequence of the following chain of equations:

$$d(\mathbf{S_w(x)}, \mathbf{S_w(y)}) = ||\mathbf{S_w(x)} - \mathbf{S_w(y)}|| =$$

$$||(\mathbf{x + w}) - (\mathbf{y + w})|| = ||\mathbf{x - y}|| = d(\mathbf{x}, \mathbf{y}).$$

3. If we combine the preceding examples with the result on composites of isometries, we see that every ***Galilean transformation*** of $\mathbf{R}^n$ having the form $\mathbf{G(x)} = \mathbf{Ax + w}$, where **A** is orthogonal, is an isometry from $\mathbf{R}^n$ to itself. It is known that products and inverses of orthogonal matrices are again orthogonal, and using this one can show that composites and inverses of Galilean transformations are also Galilean transformations (compare the proof of Proposition 4 below).

4. A standard textbook exercise in linear algebra books is to show that ***all*** isometries from $\mathbf{R}^n$ to itself are given by Galilean transformations. A proof is given at the beginning of Section 1 in the document

http://math.ucr.edu/~res/math133/metgeom.pdf

in the course directory. However, we shall not need this fact here.

Affine transformations

Before returning to geometric superposition, we shall derive some basic properties of Galilean transformations in a more general setting. The latter will include all invertible linear transformations as well as translations by fixed vectors in $\mathbf{R}^n$.

Definition. A function (or mapping) **T** from $\mathbf{R}^n$ to itself is said to be an ***affine transformation*** if it is a $1 - 1$ onto map expressible in the form $\mathbf{T(x)} = \mathbf{L(x)} + \mathbf{v}$, where **L** is an invertible linear transformation and **v** is a fixed vector in $\mathbf{R}^n$.

The requirement that **T** be $1 - 1$ and onto is actually redundant. If **L** is an invertible linear transformation and **v** is a fixed vector, then **T** as above is $1 - 1$ because $\mathbf{T(x)} = \mathbf{T(y)}$ implies $\mathbf{L(x) + v} = \mathbf{L(y) + v}$, so that $\mathbf{L(x)} = \mathbf{L(y)}$ and hence $\mathbf{x} = \mathbf{y}$ because the mapping **L** is $1-1$. Similarly, if **z** is an arbitrary vector in $\mathbf{R}^n$, then the equation **L(x) + v = z** has a solution given by $\mathbf{x} = \mathbf{L}^{-1}(\mathbf{z}-\mathbf{v})$. In particular, the preceding shows that ***every Galilean transformation is an affine transformation.***

As in the case of isometries, we begin by considering affine transformations as objects in their own right.

Proposition 4. *The identity map is an affine transformation from* $\mathbf{R}^n$ *to itself. If* **T** *is an affine transformation from* $\mathbf{R}^n$ *to itself, then so is its inverse* $\mathbf{T}^{-1}$. *Finally, if* **T** *and* **U** *affine transformations from* $\mathbf{R}^n$ *to itself, then so is their composite* $\mathbf{T}\circ\mathbf{U}$.

Proof. The first statement follows because the identity is an invertible linear transformation. To prove the second, as usual write $\mathbf{T}(\mathbf{x}) = \mathbf{L}(\mathbf{x}) + \mathbf{v}$ where **L** is invertible. Then the inverse is given by $\mathbf{T}^{-1}(\mathbf{y}) = \mathbf{L}^{-1}(\mathbf{y}) - \mathbf{L}^{-1}(\mathbf{v})$. Finally, if **U** is also an affine transformation write $\mathbf{U}(\mathbf{x}) = \mathbf{M}(\mathbf{x}) + \mathbf{w}$, where once again **M** is invertible. Then we have $\mathbf{T}\circ\mathbf{U}(\mathbf{x}) = \mathbf{L}\circ\mathbf{M}(\mathbf{x}) + \big(\mathbf{L}(\mathbf{v}) + \mathbf{w}\big)$, which shows that $\mathbf{T}\circ\mathbf{U}$ is also an affine transformation.■

Although affine transformations are not necessarily linear, they do satisfy some weak analogs of linearity.

Theorem 5. *Let* **T** *be an affine transformation from* $\mathbf{R}^n$ *to itself, and suppose that we have a barycentric linear combination*

$$\mathbf{b} = w_1\mathbf{a}_1 + w_2\mathbf{a}_2 + \ldots + w_n\mathbf{a}_n$$

where $w_1 + w_2 + \ldots + w_n = 1$. *Then* $\mathbf{T}(\mathbf{b}) = w_1\mathbf{T}(\mathbf{a}_1) + w_2\mathbf{T}(\mathbf{a}_2) + \ldots + w_n\mathbf{T}(\mathbf{a}_n)$.

Proof. Write $\mathbf{T}(\mathbf{x}) = \mathbf{L}(\mathbf{x}) + \mathbf{v}$ as usual. Then we have

$$\mathbf{T}(\mathbf{b}) = \mathbf{L}(\mathbf{b}) + \mathbf{v} = \mathbf{L}\big(w_1\mathbf{a}_1 + w_2\mathbf{a}_2 + \ldots + w_n\mathbf{a}_n\big) + \mathbf{v} =$$

$$w_1\mathbf{L}(\mathbf{a}_1) + w_2\mathbf{L}(\mathbf{a}_2) + \ldots + w_n\mathbf{L}(\mathbf{a}_n) + \big(w_1 + w_2 + \ldots + w_n\big)\mathbf{v} =$$

$$w_1\big(\mathbf{L}(\mathbf{a}_1) + \mathbf{v}\big) + w_2\big(\mathbf{L}(\mathbf{a}_2) + \mathbf{v}\big) + \ldots + w_n\big(\mathbf{L}(\mathbf{a}_n) + \mathbf{v}\big) =$$

$$w_1\mathbf{T}(\mathbf{a}_1) + w_2\mathbf{T}(\mathbf{a}_2) + \ldots + w_n\mathbf{T}(\mathbf{a}_n)$$

as required.■

This result has several important consequences.

Corollary 6. *Let* **x** *and* **y** *be distinct points in* $\mathbf{R}^n$ *and let* **T** *be an affine transformation of* $\mathbf{R}^n$ *to itself. Then for all scalars* k *we have* $\mathbf{T}(\mathbf{x} + k(\mathbf{y}-\mathbf{x})) = \mathbf{T}(\mathbf{x}) + k(\mathbf{T}(\mathbf{y}) - \mathbf{T}(\mathbf{x}))$.

Proof. We have $\mathbf{x} + k(\mathbf{y}-\mathbf{x}) = (1-k)\mathbf{x} + k\mathbf{y}$ and by the previous theorem this yields

$$\mathbf{T}(\mathbf{x} + k(\mathbf{y}-\mathbf{x})) = \mathbf{T}\big((1-k)\mathbf{x} + k\mathbf{y}\big) =$$

$$(1-k)\mathbf{T}(\mathbf{x}) + k\mathbf{T}(\mathbf{y}) = \mathbf{T}(\mathbf{x}) + k(\mathbf{T}(\mathbf{y}) - \mathbf{T}(\mathbf{x}))$$

as required.■

Corollary 7. *Let* **T** *be an affine transformation from* $\mathbf{R}^n$ *to itself. Then* **T** *takes collinear points to collinear points and noncollinear points to noncollinear points.*

Proof. Let **x, y, z** be three points in $\mathbf{R}^n$. The preceding corollary shows that if **z** lies on the line **xy,** then **T(z)** lies on the line joining **T(x)** and **T(y).** Conversely, if **T(z)** lies on the line joining **T(x)** and **T(y)** with $\mathbf{T}(\mathbf{z}) = \mathbf{T}(\mathbf{x}) + k(\mathbf{T}(\mathbf{y}) - \mathbf{T}(\mathbf{x}))$, then the derivation in the previous corollary shows that $\mathbf{T}(\mathbf{z}) = \mathbf{T}(\mathbf{x} + k(\mathbf{y} - \mathbf{x}))$. Since **T** is $1-1$ this means that $\mathbf{z} = \mathbf{x} + k(\mathbf{y} - \mathbf{x})$, so that **z** lies on the line **xy.**■

Corollary 8. *Let* **T** *be an affine transformation from* $\mathbf{R}^n$ *to itself, and let* **x** *and* **y** *be distinct points in* $\mathbf{R}^n$. *Suppose that* **a** = **T(x)** *and* **b** = **T(y).** *Then* **T** *maps the segment* **[xy]** *to the segment* **[ab]** *and* **T** *maps the ray* **[xy** *to the ray* **[ab.**

Proof. This follows from the results on affine transformations and the earlier result which determines the values of k for which $\mathbf{u} + k(\mathbf{y} - \mathbf{v})$ lies on **[uv]** or **[uv**.■

Further properties of Galilean transformations

We shall now prove a crucial result which shows that Galilean transformations have all the desired properties for rigid motions described earlier.

Theorem 9. *Every Galilean transformation* **G** *(hence every isometry) of* $\mathbf{R}^n$ *satisfies the four geometric conditions listed previously*:

1. *The function* **G** *is* $1-1$.
2. *If* **x** *and* **y** *are points of* $\mathbf{R}^n$, *then* **f** *preserves the distance between them; in other words, we have* $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{G}(\mathbf{x}), \mathbf{G}(\mathbf{y}))$.
3. *The function* **G** *sends collinear points to collinear points and noncollinear points to noncollinear points.*
4. *If* **x, y, z** *are noncollinear points of* $\mathbf{R}^n$, *then* **G** *preserves the measurement of the angle they form; in other words, we have* $|\angle \mathbf{xyz}| = |\angle \mathbf{G(x)G(y)G(z)}|$.

Proof. By the preceding discussion we may write **G** as a composite $\mathbf{T} \circ \mathbf{S}_{\mathbf{w}}$ where the factors are given as above. The first assertion follows because the difference between distinct points is positive and **G** is distance – preserving. The collinearity and noncollinearity assertions follow from general properties of affine transformations.

Finally, we must prove the result concerning angle measurements. The value $|\angle \mathbf{xyz}|$ is entirely determined by its cosine, which is equal to the quotient of $\langle \mathbf{x} - \mathbf{y}, \mathbf{z} - \mathbf{y} \rangle$ by the lengths of $\mathbf{x} - \mathbf{y}$ and $\mathbf{z} - \mathbf{y}$. If **v** = **x** or **z,** we know that the lengths of the two vectors $\mathbf{G}(\mathbf{v}) - \mathbf{G}(\mathbf{y})$ and $\mathbf{v} - \mathbf{y}$ are equal because **G** is an isometry, so it suffices to check that **G** and the inner product satisfy the following compatibility condition:

$$\langle \mathbf{x} - \mathbf{y}, \mathbf{z} - \mathbf{y} \rangle = \langle \mathbf{G}(\mathbf{x}) - \mathbf{G}(\mathbf{y}), \mathbf{G}(\mathbf{z}) - \mathbf{G}(\mathbf{y}) \rangle$$

By the factorization of **G** in the first sentence of the proof we have **G(v) = Av + w,** and it follows immediately that $\mathbf{G}(\mathbf{u}) - \mathbf{G}(\mathbf{v}) = \mathbf{A}(\mathbf{u} - \mathbf{v})$ for all **u** and **v.** Thus we may reason as before to show that

$$\langle \mathbf{G}(\mathbf{x}) - \mathbf{G}(\mathbf{y}), \mathbf{G}(\mathbf{z}) - \mathbf{G}(\mathbf{y}) \rangle = \langle \mathbf{A}(\mathbf{x} - \mathbf{y}), \mathbf{A}(\mathbf{z} - \mathbf{y}) \rangle = {}^{\mathbf{T}}(\mathbf{A}(\mathbf{x} - \mathbf{y}))\, \mathbf{A}(\mathbf{z} - \mathbf{y}) =$$

$$^{T}(\mathbf{x}-\mathbf{y})^{T}\mathbf{A}\mathbf{A}(\mathbf{z}-\mathbf{y}) \;=\; ^{T}(\mathbf{x}-\mathbf{y})\,\mathbf{I}\,(\mathbf{z}-\mathbf{y}) \;=\; \left(^{T}(\mathbf{x}-\mathbf{y})\right)\cdot(\mathbf{z}-\mathbf{y}) \;=\; \langle \mathbf{x}-\mathbf{y}, \mathbf{z}-\mathbf{y}\rangle$$

and hence **G** must preserve angle measurements.■

Geometric significance. We now have a simple method for constructing very large families of rigid motions satisfying the desired four conditions. Namely, we apply a Galilean transformation to an arbitrary subset **K** of $\mathbf{R}^n$.

Application to classical superposition

Using the results we have obtained for isometries and affine transformations, we can give a mathematical model for the sort of superposition mapping that was described earlier.

Theorem 10. *Let* **a, b, c** *be noncollinear points in* $\mathbf{R}^2$, *and let* **x, y, z** *be another triple of noncollinear points such that* $|\angle \mathbf{xyz}| = |\angle \mathbf{abc}|$. *Then there is a Galilean transformation* **G** *of* $\mathbf{R}^2$ *such that* **G(b) = y**, *the map* **G** *sends the segment* **[ba]** *into the ray* **[yx**, *and the map* **G** *sends the segment* **[bc]** *into the ray* **[yz**.

Proof. By our hypotheses the sets $\{\mathbf{a}-\mathbf{b}, \mathbf{c}-\mathbf{b}\}$ and $\{\mathbf{x}-\mathbf{y}, \mathbf{z}-\mathbf{y}\}$ are bases for $\mathbf{R}^2$. Let $M = ||\mathbf{a}-\mathbf{b}|| / ||\mathbf{x}-\mathbf{y}||$ and $N = ||\mathbf{c}-\mathbf{b}|| / ||\mathbf{z}-\mathbf{y}||$; if we set $\mathbf{u} = M(\mathbf{x}-\mathbf{y})$ and $\mathbf{v} = N(\mathbf{z}-\mathbf{y})$, then **u** and **v** are positive multiples of $\mathbf{x}-\mathbf{y}$ and $\mathbf{z}-\mathbf{y}$ such that $||\mathbf{u}|| = ||\mathbf{a}-\mathbf{b}||$ and $||\mathbf{v}|| = ||\mathbf{c}-\mathbf{b}||$. In particular, $\{\mathbf{u}, \mathbf{v}\}$ is also a basis for $\mathbf{R}^2$.

Standard results of linear algebra imply there is an invertible linear transformation **T** such that $\mathbf{T}(\mathbf{a}-\mathbf{b}) = \mathbf{u}$ and $\mathbf{T}(\mathbf{c}-\mathbf{b}) = \mathbf{v}$. We ***claim*** that **T** *is an* ***orthogonal*** *linear transformation*; *i.e.*, the mapping **T** preserves vector lengths and their squares (this is equivalent to preserving inner products). Given a vector **w** in $\mathbf{R}^2$, write

$$\mathbf{w} = p(\mathbf{a}-\mathbf{b}) + q(\mathbf{c}-\mathbf{b})$$

for suitable scalars p and q. We then have the following formulas:

$$||\mathbf{w}||^2 = p^2||(\mathbf{a}-\mathbf{b})||^2 + 2pq\langle(\mathbf{a}-\mathbf{b}), (\mathbf{c}-\mathbf{b})\rangle + q^2||(\mathbf{c}-\mathbf{b})||^2$$

$$||\mathbf{Tw}||^2 = p^2||\mathbf{T}(\mathbf{a}-\mathbf{b})||^2 + 2pq\langle\mathbf{T}(\mathbf{a}-\mathbf{b}), \mathbf{T}(\mathbf{c}-\mathbf{b})\rangle + q^2||\mathbf{T}(\mathbf{c}-\mathbf{b})||^2$$

By construction we know that $||\mathbf{T}(\mathbf{a}-\mathbf{b})|| = ||(\mathbf{a}-\mathbf{b})||$ and $||\mathbf{T}(\mathbf{c}-\mathbf{b})|| = ||(\mathbf{c}-\mathbf{b})||$. Furthermore, we also have the following:

$$\langle\mathbf{T}(\mathbf{a}-\mathbf{b}), \mathbf{T}(\mathbf{c}-\mathbf{b})\rangle = (\cos\angle\mathbf{xyz})\cdot||\mathbf{T}(\mathbf{a}-\mathbf{b})||\cdot||\mathbf{T}(\mathbf{c}-\mathbf{b})|| =$$

$$(\cos\angle\mathbf{abc})\cdot||(\mathbf{a}-\mathbf{b})||\cdot||(\mathbf{c}-\mathbf{b})|| = \langle(\mathbf{a}-\mathbf{b}), (\mathbf{c}-\mathbf{b})\rangle$$

Therefore the coefficients of p^2, $2pq$, and q^2 in the expressions for $||\mathbf{w}||^2$ and $||\mathbf{Tw}||^2$ are equal, and this implies that $||\mathbf{w}||^2 = ||\mathbf{Tw}||^2$ which implies that **T** is orthogonal by previous observations.

We now define **G** by the formula $\mathbf{G}(\mathbf{w}) = \mathbf{y} + \mathbf{T}(\mathbf{w}-\mathbf{b}) = [\mathbf{y} - \mathbf{T}(\mathbf{b})] + \mathbf{T}(\mathbf{w})$; to complete the proof, we need to verify that **G** has the properties listed in the statement of the theorem. First of all, we have $\mathbf{G}(\mathbf{b}) = \mathbf{y} + \mathbf{T}(\mathbf{0}) = \mathbf{b}$. Next, we have

$$\mathbf{G(a)} \;=\; \mathbf{y} + \mathbf{T(a-b)} \;=\; \mathbf{y} + \mathbf{u} \;=\; \mathbf{y} + M(\mathbf{x}-\mathbf{y})$$

$$\mathbf{G(c)} \;=\; \mathbf{y} + \mathbf{T(c-b)} \;=\; \mathbf{y} + \mathbf{v} \;=\; \mathbf{y} + N(\mathbf{z}-\mathbf{y})$$

so that **G(a)** and **G(c)** lie on the rays **[yx** and **[yz** respectively. By the general results on affine transformations, it follows that **[ba** and **[bc** are mapped to the rays **[y G(a) = [yx** and **[y G(c) = [yz** respectively.■

The preceding result implies a strong superposition property for congruent triangles.

<u>Proposition 11.</u> *If* △**ABC** ≅ △**XYZ**, *then there is a Galilean transformation* **G** *which maps* △**ABC** *to* △**XYZ**.

<u>Proof.</u> This is a refinement of the previous argument. Let **G** and **T** be the maps constructed above. The conditions d**(A, B)** = d**(X, Y)** and d**(B, C)** = d**(Y, Z)** imply that **T(B – A) = X – Y** and **T(C – A) = Z – Y,** and these in turn imply that **G(A) = X** and **G(C) = Z**. By the general results for affine transformations we then see that **G** must map **[AB]** to **[XY], {BC]** to **{YZ],** and **[AC]** to **[XZ].** Since the triangles under considerations are **[AB]** ∪ **[BC]** ∪ **[AC]** and **[XY]** ∪ **[YZ]** ∪ **[XZ],** it follows that **G** maps △**ABC** to △**XYZ**.■

This theorem suggests the following general definition of congruence.

<u>Definition.</u> Let **K** and **L** be subsets of $\mathbf{R}^n$. We shall say that **K** is ***congruent*** to **L** if there is a Galilean transformation **G** of $\mathbf{R}^n$ which maps **K** to **L.**

We can define affine equivalence similarly; namely, two subsets **K** and **L** are ***affinely equivalent*** if there is an affine transformation **F** of $\mathbf{R}^n$ which maps **K** to **L.** Congruent subsets are affinely equivalent, but the converse is not necessarily true. The next result illustrates this very clearly.

<u>Proposition 12.</u> *If* △**ABC** *and* △**XYZ** *are triangles in* $\mathbf{R}^2$, *then* △**ABC** *and* △**XYZ** *are affinely equivalent.*

<u>Proof.</u> Construct **G** as in the preceding corollary. Under the hypotheses we cannot conclude that **G** is a Galilean transformation, but we can conclude that it is an affine transformation, and it still maps **A, B, C** to **X, Y, Z.** As in the proof of the corollary, this gives enough information to conclude that **G** maps △**ABC** to △**XYZ**.■

<u>Remark,</u> Further study of ***affine geometry*** implies that two quadrilaterals are not necessarily affinely equivalent. For example, a parallelogram is not affinely equivalent to a proper trapezoid in which one pair of opposite sides is parallel but the other is not. We shall prove this in the next section.

<u>Origin of the term "affine."</u> (*This is just background material and it is not needed subsequently.*) As one might expect, the word "affine" comes from the same root word as does "affinity." One definition (from Webster's **1960** New World Dictionary) describes affinity as "similarity of structure, as of species of languages, implying common origin." Another contains the phrase, "resemblance in general plan or structure." This term entered geometry in the 18th and 19th century, originating in the work of L. Euler (1707 – 1783) and significantly extended in the work of A. F. Möbius (1790 – 1860). In this section we have discussed the congruence relationship between geometric figures; later in Section **III. 5** we shall discuss the weaker relationship of ***similarity***, in which the distances between corresponding points are ***proportional by a fixed constant*** rather than ***equal*** (from our viewpoint, congruence is the special case where the proportionality constant is equal to **1**). It turns out that there is a notion of ***similarity transformation*** — which is more general than a Galilean transformation but less general than an affine transformation — such that two geometric figures are similar if and only if there is a similarity transformation sending one to the other. The word "affinity" was employed to describe figures that were not necessarily similar in the usual geometrical sense but still had some common properties which distinguished them from other closely related objects. Roughly speaking, these properties involve collinearity and internal parallelism relationships. Some historical references are given below.

L. Euler, ***Intrōductio in Analysin Infīnītōrum, Tomus*** **II.** Opera Omnia, Ser. 1, Vol. 9 (ed. A .Speiser). Societās Scientārum Nātūrālium Helvēticae, Geneva, 1945. [The detailed reference is Cap. **XVIII**, artic. 442.]

A. F. Möbius, ***Der barycentrische Calcül*** (1827). Gesammelte Werke, Bd. 1 (Neudruck). Dr. M. Sändig, Wiesbaden, (West) Germany, 1967.

Geometry and geometric transformations

In this section of the notes we have seen that the notion of ***rigid motion*** or ***isometry*** plays a major role in the modern approach to geometry, and our discussion led us to more general classes of affine transformations which are closely tied to linear algebra. Such transformations will also be significant in several further contexts at later points of these notes, and in fact geometric transformations are fundamentally important to our current understanding of geometry. Most high school courses now include at least a small amount of material on geometric transformations, and some mathematicians and educators have even suggested that transformations provide the best approach to geometry; the books by Ryan and Greenberg do not quite go this far, but they do emphasize geometric transformations very systematically. The following book provides a fairly detailed account of geometry that is organized around geometric transformations.

G. E. Martin, ***Transformation Geometry: An Introduction to Symmetry*** (Undergraduate Texts in Mathematics). Springer Verlag, New York, 1982. ISBN: 0–387–90636–3.

The online file http://math.ucr.edu/~res/math133/elltangents.pdf illustrates one way in which geometrical transformations can be applied to obtain conclusions whose proofs by other methods are more difficult; in this particular case, one uses such transformations to generalize a statement about tangents and circles to the analog for ellipses.

Geometric transformations and synthetic axioms. In fact, it is possible to formulate alternative axioms for Euclidean geometry based upon the undefined concepts of point and motion (or geometric transformation), which is assumed to have some reasonable properties (the axioms). This approach to Euclidean geometry is due to G. Peano (1858 – 1932) and M. Pieri (1860 – 1913).

Appendix – Independence of the congruence axiom(s)

We have stated that it is mathematically impossible to prove the **SAS, ASA** and **SSS** congruence theorems for triangles from the other postulates introduced thus far. Over the years people have claimed many things were impossible (steamships and airplanes are particularly obvious examples), and it is easy to discount claims of impossibility as condescending or defeatist. Therefore we should explain how claims of mathematical impossibility differs from analogous claims in everyday life, and why they generally far more reliable.

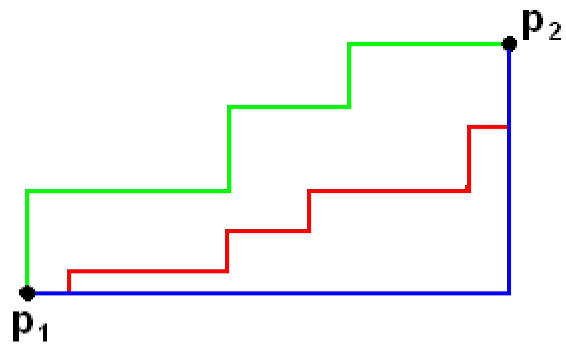
The crucial point in mathematical impossibility is that it requires one to follow logical rules very strictly and thus stay within fairly narrow limits. In contrast, consider a problem that many thought was impossible until just over **100** years ago: The building of a heavier – than – air device that could fly. One reason for the Wright brothers' breakthrough was the introduction of relatively lightweight but powerful engines that had been invented just a few years earlier. This drastically changed the rules of the game for those who wanted to build flying machines. In contrast, ***the rules of abstract logic are <u>not</u> subject to such changes.***

In principle, the idea for proving mathematical impossibility is simple; one assumes that it is possible to do something and show this leads to a logical contradiction. The ***standard elementary proof that the square root of 2 is irrational*** (which goes back to ancient Greek mathematicians) is a ***simple but typical example.*** One supposes that there is a rational number m/n whose square is equal to **2,** and one then exploits this to derive a logical contradiction (specifically, if one reduces the numerator and denominator to least terms, so that at least one is odd, then both must be even). It is not just a situation where no one has yet found a fraction of the desired type but some future genius might eventually do so; if one believes it is possible to find positive integers whose quotient is the square root of **2,** then by applying the rules of logic one ultimately ends up with a logical contradiction.

A still more basic example is given by adding two odd numbers: ***It is mathematically impossible to find two odd (positive whole) numbers whose sum is odd.*** This is true because one can prove directly that the sum of two odd numbers must be even, and it follows that no one will ever be able to find examples with the properties described in the preceding sentence.

So what do we need to do in order to prove the congruence axiom(s) cannot be derived as logical consequences of the others? It suffices to construct a ***mathematical model*** in which ***all the previous axioms are true*** but ***one of the congruence axioms is false.*** In other words, we need to construct data corresponding to points, lines, planes, linear measurement (or distance) and angular measurement which has the properties of the preceding sentence.

Our model will be very close to the usual one, the only difference being that we shall define distance in a new way. Specifically, let us take $\mathbf{R}^2$ with all the standard data except that we replace the ordinary Pythagorean metric by the so – called ***taxicab metric***: If $\mathbf{p}_1 = (x_1, y_1)$ and $\mathbf{p}_2 = (x_2, y_2)$ are points of $\mathbf{R}^2$, then the distance d_T is defined by $d_T(\mathbf{p}_1, \mathbf{p}_2) = |x_2 - x_1| + |y_2 - y_1|$.



(Source: http://mathworld.wolfram.com/TaxicabMetric.html)

The taxicab metric (sometimes also called the *Manhattan distance*) is the length of an arbitrary path connecting $\mathbf{p_1}$ and $\mathbf{p_2}$ along horizontal and vertical segments, without ever going back, like paths traveled by vehicles moving in a grid – like street pattern. It was first described explicitly by H. Minkowski (1864 – 1909) as a special case of more general phenomena involving convex sets.

We also define the ***taxicab length*** of a vector $||\mathbf{v}||_1$ to be the taxicab distance from **v** to **0.** The taxicab length has a few properties in common with the ordinary length. Here are two of them.

(1) It is nonnegative, and $||\mathbf{v}||_1 = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

(2) If k is a scalar then $||k\mathbf{v}||_1 = |k| \cdot ||\mathbf{v}||_1$.

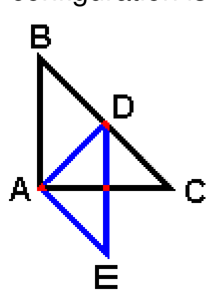
Since we are assuming that our standard model satisfies the axioms for Euclidean geometry, the key point to checking this new model satisfies the axioms from previous sections is a verification of the Ruler Postulate. This can be done by an argument similar to that in the standard model, the only difference being that the use of the Pythagorean metric and length is replaced by the taxicab metric and length; the two properties listed above imply that the new metric has enough of the properties of the usual metric that one can prove the postulates in Section 3 for the taxicab metric.

It remains to show that the new system does not satisfy the three basic congruence axioms, and to do this we need only give a pair of triangles such that $\triangle$**ABC** and $\triangle$**ADE** satisfy the SAS hypotheses

$$d_T(\mathbf{A}, \mathbf{B}) = d_T(\mathbf{A}, \mathbf{D}), \quad d_T(\mathbf{A}, \mathbf{C}) = d_T(\mathbf{A}, \mathbf{E}), \quad \text{and} \quad |\angle \mathbf{BAC}| = |\angle \mathbf{DAE}|$$

but do ***not*** satisfy $d_T(\mathbf{B}, \mathbf{C}) = d_T(\mathbf{D}, \mathbf{E})$.

One pair of such examples is given by taking $\mathbf{A} = (0, 0)$, $\mathbf{B} = (0, 2)$, $\mathbf{C} = (2, 0)$, $\mathbf{D} = (1, 1)$ and $\mathbf{E} = (1, -1)$. This configuration is depicted in the drawing below.



Since we are still using the same angle measurement, we have $|\angle \mathbf{BAC}| = |\angle \mathbf{DAE}| = 90°$. If we compute the taxicab distances between the various pairs of points we find that $d_T(A, B) = d_T(A, D) = d_T(A, C) = d_T(A, E) = 2$, but on the other hand we also have

$$d_T(B, C) = 4 > 2 = d_T(D, E).$$

Now if one could prove **SAS** from the previously stated axioms it would follow that the distances $d_T(B, C)$ and $d_T(D, E)$ would have to be equal. Since they are not, it follows that no proof of **SAS** from the earlier axioms can exist.■

In order to avoid misunderstandings, we emphasize that ***the system constructed above is not meant to be an accurate model of physical space***. It is merely an example of an abstract mathematical system which satisfies all the assumptions introduced in the previous sections of this unit.

II.5 : Euclidean parallelism

The fifth and final postulate in Euclid's ***Elements*** differs from the latter's other assumptions in several respects. All of the remaining statements are fairly simple (for example, lines can be extended indefinitely in either direction), but the last one is fairly complicated by comparison. In particular, it takes more words to state this postulate (both in English and the original Greek) than are needed for the remaining four postulates combined. For centuries scientists, philosophers and others felt it would be desirable to avoid the need for such an assumption which is so dissimilar to the others. It is also particularly noteworthy that the fifth postulate is not used until some point in Book **I** in the ***Elements***, so that a significant part of the subject is developed without this assumption. Internal evidence from the ***Elements*** and other ancient Greek writings suggest that issues related to the Fifth Postulate had concerned many mathematicians from the classical and later Greek eras.

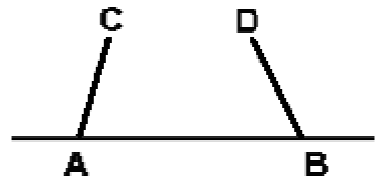
In the 5th century A. D., Proclus Diadochus (410 – 485) suggested replacing Euclid's Fifth Postulate with a simpler statement involving parallel lines; this statement is generally called *Playfair's Postulate* after J. Playfair (1748 – 1819). Although Playfair's Postulate is easier to state, it is still a more delicate assumption than the others, and there were many efforts to prove it from the remaining assumptions, beginning at least with the time of Pappus and Proclus and continuing into the 19th century. Most of these efforts either contained fatal errors or assumed some other statement, either consciously or unconsciously. By the end of the 18th century, some mathematicians had concluded that proving the Fifth Postulate from the others was futile; further work during the 19th century ultimately confirmed the impossibility of finding such a proof. *We shall discuss these matters more thoroughly in the final unit of the course.*

The main issue for now is that we need one more postulate to complete the synthetic approach to Euclidean geometry.

A modern formulation of Euclid's Fifth Postulate is probably a good place to start.

<u>EUCLID'S FIFTH POSTULATE.</u> <u>(Modern verion)</u> *Let* **AB** *be a line, and let* **C** *and* **D** *be points such that* **A**, **B**, **C** *and* **D** *are coplanar and both* **C** *and* **D** *lie on the same side of* **AB.** *Then the open rays* **(AC** *and* **(BD** *have a point in common if and only if*

$$|\angle \mathbf{CAB}| + |\angle \mathbf{DBA}| < 180°.$$



In fact, one can derive the "only if" implication from the other assumptions, so the "if" implication is the central point. The picture shows a situation in which the angle measurement inequality holds, and it seems clear that one could extend the two rays to find a point where they both meet. If the sum of the angle measurements were greater, then it is possible that the point of intersection would not lie on the printed page, and if the sum is close enough to $180°$ any point may be too far away to be located by any ordinary physical means.

Before introducing the statement suggested by Proclus and Playfair, we shall need to formalize the concept of parallelism.

<u>Definition.</u> Given two lines **L** and **M** in the plane or space, we say that **L** and **M** are ***parallel*** (written **L || M**) if **L** and **M** are coplanar but have no points in common.

If we are working in the plane, the condition that **L** and **M** be coplanar is unnecessary, but there are examples of lines in space which are both noncoplanar and disjoint; such pairs are called ***skew lines.*** There are many simple examples of skew lines, and here is one of them: Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the standard basis of $\mathbf{R}^3$ given by unit vectors, let **L** be the line consisting of all points expressible as $u\mathbf{e}_1$ for some scalar u, and let **M** be the line consisting of all points expressible as $\mathbf{e}_2 + v\mathbf{e}_3$ for some scalar v. The verification that these lines are disjoint and not coplanar is left as an exercise.

We are now ready to give the postulate on parallel lines which is equivalent to Euclid's Fifth Postulate.

<u>Axiom P – 0</u> <u>(*Playfair's Postulate*)</u>**:** *Given a line* **L** *and a point* **X** *which is not on* **L**, *there is a unique line* **M** *such that* **X** $\in$ **M** *and* **L || M**.

As in the case of Euclid's Fifth Postulate, one can get by with a slightly weaker assumption. Using the axioms presented in previous sections one can always construct at least one parallel to **L** through **X**; for example, this can be done by dropping a perpendicular **N** from **X** to **L** and then constructing a perpendicular **M** to **N** at **X** in the plane determined by the intersecting lines **L** and **N.** One can then prove that **L** and **M** have no points in common. Thus the real question answered by **P – 0** is ***whether more than one parallel through* X *can exist,*** and by this postulate the answer is negative.

Algebraic interpretation

In $\mathbf{R}^2$ and $\mathbf{R}^3$ there is a simple characterization of parallel lines in terms of linear algebra.

Proposition 1. *Let* **x + V** *be a line in* $\mathbf{R}^2$ *and* $\mathbf{R}^3$, where **V** *is a* $1-$ *dimensional vector subspace spanned by the nonzero vector* **v**, *and suppose that* **z** *is a vector not on* **x + V**. *Then the lines* **x + V** *and* **z + V** *are parallel.*

Proof. The condition on **z** implies that **z – x** does not lie in **V**. Therefore it follows that **v** and **z – x** are linearly independent. Let **U** be the $2-$ dimensional vector subspace they span. Clearly **x + V** is contained in **x + U** by construction, and if **z +** k**v** is a typical vector in the line **z + V** then the expansion **z +** kv = **x + (z – x)_+** k**v** expresses the left hand side as a sum of **x** and two vectors in **U**, and consequently the vector in question lies in **x + U**. Thus the lines **x + V** and **z + V** are coplanar; it remains to show that they have no points in common. Assume the contrary, and suppose that w is a common point. Then there are scalars p and q such that

$$\mathbf{x} + p\mathbf{v} \;=\; \mathbf{w} \;=\; \mathbf{z} + q\mathbf{v} \;=\; \mathbf{x} + (\mathbf{z} - \mathbf{x}) + q\mathbf{v}$$

and if we subtract **x** from both sides and rearrange terms we obtain **z – x** = $(p-q)$**v**, which in turn implies that **z – x** lies in **V**. This contradicts a previous observation. The source of the contradiction is our assumption that **x + V** and **z + V** had a point in common, so this must be false and the lines in question must be parallel.■

We can use this to give a very simple proof of the next result, which is pretty obvious in the plane (look at a piece of ruled notebook paper) but less so in $3-$ dimensional space.

Theorem 2. *Let* **L**, **M** *and* **N** *be lines in* $\mathbf{R}^2$ *or* $\mathbf{R}^3$ *such that* **L || M** *and* **M || N**. *Then either* **L = N** *or else* **L || N**.

In other words, ***if two distinct lines are parallel to a third line, then they are parallel to each other.***

Proof. Write the line **M** as **x + V** where **V** is a $1-$ dimensional vector subspace. Then it follows that **L = a + V** and **N = b + V** where neither **a** nor **b** belongs to **M.** If **a** also does not belong to **N,** then by the previous reasoning we know that **L** and **N** are parallel. On the other hand, if **a** does belong to **N** then we may write **a = b +** k**v** where **v** is a nonzero vector spanning **V** and k is some scalar. Therefore an arbitrary vector in **L** has the form **a +** q**v = a = b +** k**v +** q**v,** which implies that it also lies in **N = b + V.** In particular, this means that **L** is contained in **N.** Since there is only one line containing two points and **L** contains at least two points, this implies **L = N.**■

Parallelism and affine transformations

At the end of the previous section we mentioned that a parallelogram and a proper trapezoid are not affinely equivalent. We can use the material of this section to prove a general result which will imply our earlier assertion.

Theorem 3. *Let* **a, b, c, d** *be four points in* $\mathbf{R}^2$ *such that no three are collinear, and let* **T** *be an affine transformation of* $\mathbf{R}^2$ *to itself. Then the lines* **ab** *and* **cd** *are parallel if and only if the lines* **T(a)T(b)** *and* **T(c)T(d)** *are parallel.*

Since every Galilean transformation is an affine transformation, it follows immediately that ***Galilean transformations send parallel lines to parallel lines.***

Proof. As usual write **T(x) = L(x) + v** where **L** is an invertible linear transformation and **v** is a fixed vector. If **ab || cd**, then we may write **ab = a + W** and **cd = c + W** where **W** is some 1 – dimensional vector subspace. It follows that both **b – a** and **d – c** lie in **W,** and hence these nonzero vectors must be nonzero scalar multiples of each other, say **d – c =** k**(b – a).** Similarly, we have that the lines **T(a)T(b)** and **T(c)T(d)** are respectively given by all vectors of the forms

$$\mathbf{T(a)} + p(\,\mathbf{T(b)} - \mathbf{T(a)}\,) \qquad\qquad \mathbf{T(c)} + q(\,\mathbf{T(d)} - \mathbf{T(c)}\,)$$

for suitable scalars p and q. Since earlier considerations imply that **T(c)** does not lie on **T(a)T(b),** it will suffice to show that **T(d) – T(c)** is a nonzero multiple of **T(b) – T(a).** Using the description of **T** in the first sentence we have

$$\mathbf{T(d) - T(c)} = \mathbf{(L(d) + v)} - \mathbf{(L(c) + v)} = \mathbf{L(d) - L(c)} = \mathbf{L(d - c)} =$$

$$\mathbf{L(}k\mathbf{(b - a))} = k\,\mathbf{L(b - a)} = k\big(\mathbf{(L(b) + v)} - \mathbf{(L(a) + v)}\big) = k(\,\mathbf{T(b) - T(a)}\,)$$

and therefore **T(d) – T(c)** is indeed a nonzero multiple of **T(b) – T(a)** as required.

Conversely, if the lines **T(a)T(b)** and **T(c)T(d)** are parallel, then **T(d) – T(c)** is a nonzero multiple of **T(b) – T(a),** so we have an equation of the form

$$\mathbf{T(d) - T(c)} = k\mathbf{(T(b) - T(a))}$$

for a suitable scalar k. We then have

$$\mathbf{L(}\,k\mathbf{(b - a)}\,) = k\,\mathbf{L(b - a)} = k\big(\mathbf{(L(b) + v)} - \mathbf{(L(a) + v)}\big) = k\mathbf{(T(b) - T(a))} =$$

$$\mathbf{T(d) - T(c)} = \mathbf{(L(d) + v)} - \mathbf{(L(c) + v)} = \mathbf{L(d) - L(c)} = \mathbf{L(d - c)}.$$

Since **L** is invertible, this means that k(**b – a**) must be equal to **d – c,** and therefore the lines **ab** and **cd** must be parallel.■

Corollary 4. *Let* **a, b, c, d** *be four points in* $\mathbf{R}^2$ *such that no three are collinear, and let* **T** *be an affine transformation of* $\mathbf{R}^2$ *to itself. If* **ab** *and* **cd** *are parallel and* **ad** *and* **bc** *are parallel, then* **T(a)T(b)** *and* **T(c)T(d)** *are parallel and* **T(a)T(d)** *and* **T(b)T(c)** *are parallel.*■

In particular, the corollary and results of the previous section show that the image of the parallelogram

$$\mathbf{[ab]} \cup \mathbf{[bc]} \cup \mathbf{[cd]} \cup \mathbf{[ad]}$$

is also a parallelogram and hence cannot be a proper trapezoid. This proves an assertion at the end of the previous section.■

In the previous section we mentioned that the Galilean transformations of $\mathbf{R}^n$ could be characterized as the $\mathbf{1-1}$ correspondences from $\mathbf{R}^n$ to itself that preserve distances between points (***isometries***). There is an analogous synthetic characterization of affine transformations in terms of mappings called ***collineations***; in the planar case, these are $\mathbf{1-1}$ correspondences from $\mathbf{R}^2$ to itself which take collinear subsets to collinear subsets and noncollinear subsets to noncollinear subsets. Appendix $\mathbf{E}$ of Ryan contains a proof that collineations and affine transformations of $\mathbf{R}^2$ are the same (also see pages 39 – 40 of that book for background).

Incidentally, a similar result holds for $\mathbf{3}-$ dimensional space; in this case the definition of collineation must be modified to say that the $\mathbf{1-1}$ correspondence also takes coplanar subsets to coplanar subsets and noncoplanar subsets to noncoplanar subsets.

Appendix – Coordinate affine spaces

One obvious feature of Playfair's Postulate is that it makes no reference to linear or angular measurement. This in itself strongly suggests that ***questions about parallel lines can be studied, at least to some extent, in their own right and independently from any questions about measurement.*** In fact, one can go quite far in this direction, and it leads to significant insights into questions of independent interest. Examples include the finite geometries that were mentioned in Section $\mathbf{1}$ of this unit.

A further example of the power of Playfair's Postulate is a fundamental ***coordinatization theorem*** for abstract systems $(\mathbf{S}, \mathcal{L}, \mathcal{P})$ of points, lines and planes which satisfy both the $\mathbf{3}-$ dimensional Incidence Axioms in Section $\mathbf{1}$ and Playfair's Postulate (such a system is often called an ***affine*** $\mathbf{3}-$ ***space***). The statement of this result requires a description of certain basic systems which satisfy the conditions in the preceding sentence.

The first step is a fundamental fact about linear algebra; namely, ***everything at the beginning of the subject*** about subspaces, bases, linear transformations, matrices, and so on ***through the theory of determinants will remain valid if even if we do not work over the real or complex numbers.*** All that one needs is a system with notions of *addition*, *subtraction*, *multiplication*, and *division* (by nonzero quantities) that satisfy the usual algebraic properties. The real and complex numbers are examples of such systems, and the rational numbers are another. Furthermore, many of the "clock arithmetic" systems, in which one identifies two integers whose difference is equal to a multiple of some fixed positive integer $k > 1$, are also allowable choices for scalars; specifically, this is true if (and only if) k is a prime. Many linear algebra texts explicitly note this level of generality, and in nearly every text it is at least implicit (see also the online file http://math.ucr.edu/~res/progeom/pgnotesappa.pdf).

If **F** is a system which satisfies the given conditions on the four basic arithmetic operations, then for each positive integer n we can define the $n-$ dimensional vector space $\mathbf{F}^n$ in which addition and multiplication are defined coordinatewise, exactly as in the real or complex case. We can also define translates of subspaces exactly as in Unit $\mathbf{I}$ of these notes, and we can take the sets $\mathcal{L} = \mathcal{L}(\mathbf{F}^n)$ and $\mathcal{P} = \mathcal{P}(\mathbf{F}^n)$ of lines and planes to be the translates of $1-$dimensional and $2-$dimensional vector subspaces. One can then extend the previous arguments to prove the following result:

Theorem 5. *Let $n = 2$ or 3, and let $(\mathbf{F}^n, \mathcal{L}, \mathcal{P})$ be as above. Then $(\mathbf{F}^n, \mathcal{L}, \mathcal{P})$ satisfies the relevant Incidence Axioms (the first two if $n = 2$, all of them if $n = 3$) and Playfair's Postulate.*■

One can even go a little further and make nearly everything work for scalars that do not necessarily satisfy the commutative multiplication identity $ab = ba$; the most notable exception is that the theory of determinants cannot be extended to this setting. The most widely known and used example of this sort is given by the algebra $\mathbf{H}$ of ***quaternions,*** which corresponds to a notion of multiplication on $\mathbf{R}^4$; the notation reflects the fact that this algebra was first discovered and publicized by W. R. Hamilton (1805 – 1865), who is also known for numerous other contributions to mathematics and physics. Some online references for the quaternions are given below:

http://en.wikipedia.org/wiki/Division_ring

http://en.wikipedia.org/wiki/Field_theory_(mathematics)

http://en.wikipedia.org/wiki/Quaternion

http://en.wikipedia.org/wiki/Frobenius_theorem_(real_division_algebras)

We shall not discuss the quaternions explicitly in this course, but more detailed information on $\mathbf{H}$ appears in the following reference:

I. N. Herstein, ***Topics in Algebra*** (2nd Ed.). Wiley, New York, 1975.
ISBN: 0–571–01090–1.

Algebraic systems which satisfy all the standard properties for addition, subtraction, multiplication and division except perhaps the commutative law of multiplication are called ***division rings*** or ***skew – fields***; if the commutative law of multiplication holds, the system is called a ***field.*** If **F** is a division ring and n is a positive integer, then we can make $\mathbf{F}^n$ into a ***right*** or ***left vector space*** over **F** as before. We have to be careful about the difference between left and right multiplication because the commutative law of multiplication fails; for example, we need to distinguish ***between right vector subspaces*** and ***left vector subspaces,*** and when defining lines and planes we much make a choice of whether we want to use translates of right or left vector subspaces. However, if we systematically take these issues into account, we obtain the following generalization of the previous result.

Theorem 6. *Let $n = 2$ or 3, and let $(\mathbf{F}^n, \mathcal{L}, \mathcal{P})$ be as above, where **F** is a skew – field. Then $(\mathbf{F}^n, \mathcal{L}, \mathcal{P})$ satisfies the relevant Incidence Axioms (the first two if $n = 2$, all of them if $n = 3$) and Playfair's Postulate.*■

Notation. We shall say that the system associated to ***right vector subspaces*** is the *standard coordinate model for affine n – space over the skew – field* **F.**

In the $3-$dimensional case there is a remarkable converse to Theorem 6:

Theorem 7. (Coordinatization Theorem) *Let* **(S, $\mathcal{L}$, $\mathcal{P}$)** *be a system of points, lines and planes which satisfy both the* **3 –** *dimensional Incidence Axioms in Section* **1** *and Playfair's Postulate. Then there is a* **1 – 1** *correspondence* h *from* **S** *onto some coordinate affine* **3 –** *space* $\mathbf{F}^3$ *such that a subset* **A** *of* **S** *is a line in* **S** *if and only if its image* h**[A]** *under* h *is a line in* $\mathbf{F}^3$*, and a subset* **C** *of* **S** *is a plane in* **S** *if and only if its image* h**[C]** *under* h *is a plane in* $\mathbf{F}^3$**.**

In other words, ***every* 3 *– space satisfying Playfair's Postulate is mathematically equivalent to one of the coordinate models*** we have described above.■

The crucial ideas in the proof of this result are due to K. F. von Staudt (1798 – 1867). More detailed information (formulated using concepts from Unit **IV** of these notes) appears in the books cited below:

> W. V. D. Hodge and D. Pedoe, ***Methods Of Algebraic Geometry, Volume* I.** Cambridge Univ. Press, Cambridge (U. K.) and New York, 1968. ISBN: 0–521–46901–5. [The specific reference is Chapter **VI.**]
>
> J. A. Murtha and E. R. Willard, ***Linear Algebra and Geometry.*** Holt, Rinehart and Winston, New York, 1969. ISBN: 0–030–74485–7. [The specific reference is Sections 4.6 and 4.7.]

In contrast to the preceding coordinatization theorem, the situation for planes is more complicated. and in fact there exist affine planes which are not equivalent to any of the coordinate models.

> ***Coordinatization for affine planes.*** (*This is additional information at an advanced level and will not be needed elsewhere in the notes.*) Questions about coordinatizing affine planes are generally discussed using the notions of projective geometry described in Unit **IV** of these notes. Specifically, given an affine plane there is an associated projective plane obtained by adding an extra line of points at infinity, and conversely given a projective plane and a line in the latter, one can obtain an affine plane by removing the given line. There is a corresponding notion of coordinates for projective planes, and the coordinatization of an affine plane turns out to be equivalent to the coordinatization of the associated projective plane. For projective planes, these issues are discussed in the following online reference which will serve as the basis for our discussion:
>
> http://math.ucr.edu/~res/progeom/pgnotes04.pdf
>
> An example of an affine plane which cannot be coordinatized is given by taking the example on page **73** of this reference and removing the line of points at infinity. Further examples are given by removing one line from each of the examples mentioned in the first paragraph of page **74.** As noted on pages **84 – 86** of the same document, the methods which yield Theorem **7** can be extended to give a sufficient condition for the coordinatization of an affine plane. The condition is an assumption that the statement of Desargues' Theorem, which is discussed in sections **IV.1** and **IV.5** of these notes, is true in the associated projective plane.

Final remark. One natural question about the coordinatization theorem is whether there is an intrinsically geometric condition that is equivalent to the commutativity of multiplication in the skew – field **F.** In fact, there is such a condition; namely, a suitable analog of the Pappus Hexagon Theorem in Unit **IV.5** of the notes must be true in the abstract geometrical system **(S, $\mathcal{L}$, $\mathcal{P}$).**