Irreducibility of affine and projective planes

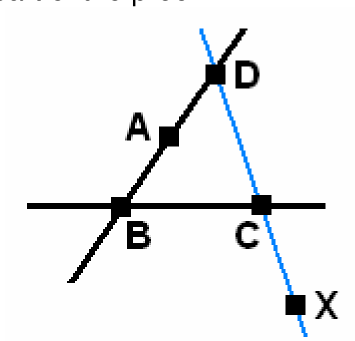
The document http://math.ucr.edu/~res/math133/nonmetric_models.pdf states and proves the theorem stated below; our purpose here is to give some drawings which may be helpful for understanding the proof.

Theorem. *Suppose that* **P** *is either a finite projective plane or a finite affine plane which contains more than* 4 *points (hence every line contains at least* 3 *points).* *If* **Q** *is a flat, noncollinear subset of* **P**, *then* **Q** = **P.**

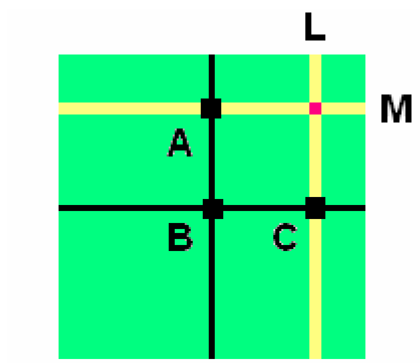
Recall that a subset **Q** of **P** is ***flat*** if it has the following property:

If **A, B** $\in$ **Q,** *then* ***all*** *points of the line* **AB** *are contained in* **Q.**

The proof of the theorem has two main parts, the first of which suffices for projective planes and the second of which deals with possibilities that are unique to the affine case. Here is a drawing for the first part of the proof:



We are given that the noncollinear points **A, B, C** lie in the flat subset **Q.** If **X** is a point such that the line **XC** meets **AB** at some point (which is always the case in a projective plane), then **X** must also lie in **Q.** This proves the result for projective planes and also yields crucial information for affine planes.



More precisely, for an affine plane the preceding observation implies that **Q** contains all points of **P** except possibly those on the line **L** through **C** which is parallel to **AB.** The same considerations show that **Q** also contains all points of **P** except possibly those on the line **M** through **A** which is parallel to **BC,** and thus **Q** contains all points of **P** except perhaps the point **E** where **L** and **M** intersect. Our assumptions ensure that every line has at least three points, and therefore there are at least two points on **L** besides **E.** We know that these points lie in **Q,** and this implies that **E** must also lie in **Q,** which means that **Q** contains every point of **P.**■