Exercises for Unit II
(Vector algebra and Euclidean geometry)

II.1 : Approaches to Euclidean geometry

1. What is the minimum number of planes containing three concurrent noncoplanar lines in coordinate 3 – space $\mathbf{R}^3$?

2. What is the minimum number of planes in coordinate 3 – space $\mathbf{R}^3$ containing five points, no four of which are coplanar? [**_Hint:_** No three of the points are collinear.]

3. Suppose that **a, b, c, d** are four noncoplanar points in $\mathbf{R}^3$. Explain why the lines **ab** and **cd** are disjoint but not coplanar (in other words, they form a pair of **_skew lines_**).

4. Let **L** be a line in coordinate 2 – space $\mathbf{R}^2$ or 3 – space $\mathbf{R}^3$, let **x** be a point not on **L,** and let $\mathbf{p}_1, \ldots, \mathbf{p}_n$ be points on **L.** Prove that the lines $\mathbf{x}\mathbf{p}_1, \ldots, \mathbf{x}\mathbf{p}_n$ are distinct. Why does this imply that $\mathbf{R}^2$ and $\mathbf{R}^3$ contain infinitely many lines?

5. Suppose that $\mathbf{L}_1, \ldots, \mathbf{L}_n$ are lines in coordinate 2 – space $\mathbf{R}^2$ or 3 – space $\mathbf{R}^3$. Prove that there is a point **q** which does not lie on any of these lines. [**_Hint:_** Take a line **M** which is different from each of $\mathbf{L}_1, \ldots, \mathbf{L}_n$; for each j we know that **M** and $\mathbf{L}_j$ have at most one point in common, but we also know that **M** has infinitely many points.]

6. Suppose that $\mathbf{p}_1, \ldots, \mathbf{p}_n$ are points in coordinate 2 – space $\mathbf{R}^2$ or 3 – space $\mathbf{R}^3$. Prove that there is a line **L** which does not contain any of these points. [**_Hint:_** Let **x** be a point not equal to any of $\mathbf{p}_1, \ldots, \mathbf{p}_n$ and take a line **L** through **x** which is different from each of $\mathbf{x}\mathbf{p}_1, \ldots, \mathbf{x}\mathbf{p}_n$.]

7. Let **P** be a plane in coordinate 3 – space $\mathbf{R}^3$, let **a, b, c** be noncolinear points on **P,** let **z** be a point which is not on **P,** and let **u** and **v** be distinct points on the line **cz.** Show that the planes **P, abu** and **abv** are distinct. Using this and ideas from Exercise **4,** prove that there are infinitely many planes in coordinate 3 – space $\mathbf{R}^3$ which contain the line **ab.**

8. Suppose we have an abstract system **(P, $\mathcal{L}$)** consisting of a set **P** whose elements we call **points** and a family of proper subsets $\mathcal{L}$ that we shall call **lines** such that the points and lines satisfy axioms $(\mathbf{I} - \mathbf{1})$ and $(\mathbf{I} - \mathbf{2})$. Assume further that every line **L** in **P** contains at least three points. Prove that **P** contains at least seven points.

> **Remark:** The notes give an example of a system **(P, $\mathcal{L}$)**
> such that that every line in **P** contains **_exactly_** three points
> and **P** itself contains **_exactly_** seven points.

II.2 : Synthetic axioms of order and separation

1. Suppose that **A, B, C** are collinear points in $\mathbf{R}^3$ whose coordinates are given by (a_1, a_2, a_3), (b_1, b_2, b_3), and (c_1, c_2, c_3) respectively. Prove that **A**∗**B**∗**C** holds if we have $a_1 < b_1 < c_1$, $a_2 = b_2 = c_2$, and $a_3 > b_3 > c_3$.

2. In the coordinate plane $\mathbf{R}^2$, let **A** $= (1, 0)$, **B** $= (0, 1)$, **C** $= (0, 0)$ and **D** $= (-2, -1)$. Show that the lines **AB** and **CD** intersect in a point **X** such that **A**∗**X**∗**B** and **D**∗**C**∗**X** hold. [*Hint:* Construct **X** explicitly.]

3. In the coordinate plane $\mathbf{R}^2$, suppose that **A, B, C, D, E** are points not on the same line such that **A**∗**B**∗**C** and **A**∗**D**∗**E** hold. Prove that the segments **(BE)** and **(CD)** have a point in common. [*Hint:* Use barycentric coordinates with respect to the points **A, B** and **D**.]

4. In the coordinate plane $\mathbf{R}^2$, let **A** $= (4, -2)$ and **B** $= (6, 8)$, and let **L** be the line defined by the equation $4y = x + 10$. Determine whether **A** and **B** lie on the same side of **L**.

5. In the coordinate plane $\mathbf{R}^2$, let **A** $= (8, 5)$ and **B** $= (-2, 4)$, and let **L** be the line defined by the equation $y = 3x - 7$. Determine whether **A** and **B** lie on the same side of **L**.

6. Let **L** be a line in the plane **P,** and suppose that **M** is some other line in **P** such that **L** and **M** have no points in common. Prove that all points of **M** lie on the same side of **L** in **P**.

7. State and prove a generalization of the previous result to disjoint planes in $3-$space.

8. Suppose that **L** is a line in the plane of triangle △**ABC.** Prove that **L** cannot meet all three of the open sides **(AB), (BC)** and **(AC).**

9. Let $\mathbf{V}$ be a vector space over the real numbers, and let $\mathbf{S} = \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a subset of $\mathbf{V}$. A vector $\mathbf{x}$ in $\mathbf{V}$ is said to be a ***convex combination*** of the vectors in $\mathbf{S}$ if $\mathbf{x}$ is expressible as a linear combination of the form $a_0\mathbf{v}_0 + a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n$ such that $a_0 + a_1 + \ldots + a_n = 1$ and $0 \le a_j \le 1$ for all j. Let $\mathbf{S} = \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a subset of $\mathbf{V}$, and let $\mathbf{T} = \{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be a set of vectors in $\mathbf{V}$ which are convex combinations of the vectors in $\mathbf{S}$. Suppose that $\mathbf{y}$ is a vector in $\mathbf{V}$ which is a convex combination of the vectors in $\mathbf{T}$. Prove that $\mathbf{y}$ is also a convex combination of the vectors in $\mathbf{S}$.

10. For which values of t is the point $(17, t)$ on the same side of the line defined by the equation $2x + 3y = 6$ as the point $(4, 5)$?

11.   Given the three points $(2, 5), (6, 5)$ and $(6, 2),$ find two of them which lie on the same side of the line defined by the equation $3y - 2x = 1.$
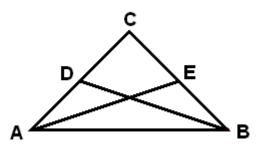
II.3 : Measurement axioms

1.   Suppose we are given a line containing the two points **A** and **B.** Then every point **X** on the line can be expressed uniquely as a sum $\mathbf{A} + k(\mathbf{B} - \mathbf{A})$ for some real number $k.$ Let $\mathbf{f} : \mathbf{L} \to \mathbf{R}$ be defined by $\mathbf{f(X)} = k\,d(\mathbf{A}, \mathbf{B}).$ Prove that **f** defines a $1 - 1$ correspondence such that $d(\mathbf{X}, \mathbf{Y}) = |\mathbf{f(X)} - \mathbf{f(Y)}|$ for all points **X, Y** on **L .** [**_Hint:_** Recall that $d(\mathbf{X}, \mathbf{Y}) = |\mathbf{X} - \mathbf{Y}|.$]

2.   Suppose that we are given a line **L** and **_two_** distinct $1 - 1$ correspondences between **L** and the real line **R** which satisfy the condition in the Ruler Postulate $\mathbf{D} - \mathbf{3},$ say $\mathbf{f} : \mathbf{L} \to \mathbf{R}$ and $\mathbf{g} : \mathbf{L} \to \mathbf{R}.$ Prove that these functions satisfy a relationship of the form $\mathbf{g(X)} = a\,\mathbf{f(X)} + b,$ where a and b are real numbers with $a = \pm 1.$ [**_Hint:_** Look at the function $\mathbf{h} = \mathbf{g} \circ \mathbf{f}^{-1},$ which is a distance $-$ preserving $1 - 1$ correspondence from **R** to itself. Show that such a map has the form $\mathbf{h}(t) = at + b,$ where a and b are as above. To do this, first show that if **h** is a distance $-$ preserving $1 - 1$ correspondence from **R** to itself then so is $\mathbf{k}(t) = \mathbf{h}(t) - \mathbf{h}(0),$ then show that **k** must be multiplication by $\pm 1.$]

3.   In the coordinate plane, determine whether the point $\mathbf{X} = (9, 4)$ lies in the interior of $\angle\mathbf{ABC},$ where $\mathbf{A} = (7, 10), \mathbf{B} = (2, 1)$ and $\mathbf{C} = (11, 1).$ Also, determine the values of k for which $(17, k)$ lies in the interior of $\angle\mathbf{ABC}.$

4.   Answer the same questions as in the preceding exercise for $\mathbf{X} = (30, 200)$ and $\mathbf{X} = (75, 135).$

5.   Answer the same questions as in Exercise 3 when $\mathbf{X} = (-5, 8),$ $\mathbf{A} = (-4, 8),$ $\mathbf{B} = (-1, 2)$ and $\mathbf{C} = (-5, -12).$

6.   Suppose we are given $\angle\mathbf{ABC}.$ Prove that the open segment **(AC)** is contained in the interior of $\angle\mathbf{ABC}.$

7.   Suppose we are given $\triangle\mathbf{ABC}$ and a point **D** in the interior of this triangle, and let **E** be any point in the same plane except **D.** What general conclusion about the intersection of $\triangle\mathbf{ABC}$ and **[DE** seems to be true? Illustrate this conjecture with a rough sketch.

8.   Suppose that we are given $\triangle\mathbf{ABC}$ and a point **X** on **(BC).** Prove that the open segment **(AX)** is contained in the interior of $\triangle\mathbf{ABC}.$

9.   Let $\angle\mathbf{ABC}$ be given. Prove that there is a point **D** on the same plane such that **D** and **A** lie on opposite sides of **BC,** but $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{D}, \mathbf{B})$ and $|\angle\mathbf{ABC}| = |\angle\mathbf{DBC}|.$ [**_Hint:_** This uses both the Ruler and Protractor Postulates.]

10.   Suppose we are given △**ABC** and a point **D** in the interior of this triangle. Prove that **D** lies on an open segment **(XY),** where **X** and **Y** lie on the triangle.

II.4 : Congruence, superposition and isometries

1.   Let ∠**ABC** be given. Prove that there is a unique *__angle bisector__* ray **[BD** such that **(BD** is contained in the interior of ∠**ABC** and |∠**ABD**| = |∠**DBC**| = ½ |∠**ABC**|. [*__Hints:__* Let **E** be the unique point on **(BA** such that $d(\mathbf{B}, \mathbf{E}) = d(\mathbf{B}, \mathbf{C}),$ and let **D** be the midpoint of **[CE].** Recall that there should be proofs for both existence and uniqueness.]

2.   Give an example of a triangle △**ABC** for which the standard formal congruence statement △**ABC** ≅ △**BCA** is false.

3.   Suppose we are given △**ABC** and △**DEF** such that △**ABC** ≅ △**DEF.** Explain why we also have △**ACB** ≅ △**DFE** and △**BCA** ≅ △**EFD.**

4.   Suppose △**ABC** is an isosceles triangle with $d(\mathbf{A}, \mathbf{C}) = d(\mathbf{B}, \mathbf{C}),$ and let **D** and **E** denote the midpoints of **[AC]** and **[BC]** respectively. Prove that △**DAB** ≅ △**EBA.**

5.   Suppose we are given △**ABC** and △**DEF** such that △**ABC** ≅ △**DEF,** and suppose that we have points **G** on **(BC)** and **H** on **(EF)** such that **[AG** and **[DH** bisect ∠**BAC** and ∠**EDF** respectively. Prove that △**GAB** ≅ △**HDE.**

6.   Conversely, in the setting of the previous exercise suppose that we are not given the condition △**ABC** ≅ △**DEF,** but we are given that △**GAB** ≅ △**HDE.** Prove that △**ABC** ≅ △**DEF.**

7.   Let **K** be a convex subset of $\mathbf{R}^n.$ A point **X** in **K** is said to be an *__extreme point__* of **K** if it is not between two other points of **K.** Suppose that **T** is an affine transformation of $\mathbf{R}^n,$ and suppose that **T** maps the convex set **K** onto the convex set **L.** Prove that **T** maps the extreme points of **K** to the extreme points of **L.**

8.   Suppose that **T** is the affine transformation of $\mathbf{R}^n$ given by **T(x)** = **L(x) + v,** where **L** is an invertible linear transformation and **v** is a fixed vector in $\mathbf{R}^n.$ If **L** is given

by the $n \times n$ matrix $\mathbf{A} = (a_{i,j})$ and $\mathbf{v} = (b_1, \dots, b_n)$, then one has the following expression for $\mathbf{y} = \mathbf{T(x)}$ in terms of coordinates:

$$y_i = a_{i,1} x_1 + \dots + a_{i,n} x_n + b_i$$

Let **DT** be the **derivative matrix** whose (i, j) entry is given by

$$\frac{\partial y_i}{\partial x_j}$$

(a)   Show that the (i, j) entry of **DT** is equal to $a_{i,j}$.

(b)   If $\mathbf{T_1}$ and $\mathbf{T_2}$ are affine transformations of $\mathbf{R}^n$, explain why $\mathbf{D(T_1 \circ T_2)}$ is the matrix product $\mathbf{D(T_1) D(T_2)}$. [**_Hint:_** Expand the composite $\mathbf{T_1 \circ T_2}$.]

(c)   Explain why **T** is a translation if and only if **D(T)** is the identity matrix.

(d)   If **T** is a translation and **S** is an arbitrary affine transformation, prove that the composite $\mathbf{S^{-1} \circ T \circ S}$ is a translation. What is its value at the vector $\mathbf{0}$?

9.   The **_vertical reflection_** $\mathbf{S}(c)$ about the horizontal line $y = c$ in $\mathbf{R}^2$ is the affine map defined by $(x_1, x_2) = (x_1, 2c - x_2)$. Show that $\mathbf{S}(c)$ sends the horizontal line $y = c$ into itself and interchanges the horizontal lines $y = 2c$ and $y = 0$. Prove that the composite of two vertical reflections $\mathbf{S}(a)\mathbf{S}(b)$ is a translation, and the composite of three vertical reflections $\mathbf{S}(a)\mathbf{S}(b)\mathbf{S}(c)$ is a vertical reflection $\mathbf{S}(d)$; evaluate d explicitly. [**_Hints:_** For the twofold composite, what is the derivative matrix? Also, explain why the twofold composite $\mathbf{S}(a)\mathbf{S}(b)$ sends $(0, 0)$ to a point whose first coordinate is equal to zero.]

10.   Let **A** be the orthogonal matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

where θ is a real number. Show that there is an orthonormal basis $\{\mathbf{u}, \mathbf{v}\}$ for $\mathbf{R}^2$ such that $\mathbf{Au} = \mathbf{u}$ and $\mathbf{Av} = -\mathbf{v}$.

11.   Let **A** be the orthogonal rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where θ is a real number which is **_not_** an integral multiple of 2π, let $\mathbf{b} \in \mathbf{R}^2$, and let **T** be the Galilean transformation $\mathbf{T(x)} = \mathbf{Ax} + \mathbf{b}$. Prove that there is a unique vector $\mathbf{z}$ such that $\mathbf{T(z)} = \mathbf{z}$. [**_Hint:_** **A** is not the identity matrix, and in fact $\mathbf{A} - \mathbf{I}$ is invertible; prove the latter assertion.]

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II.5 : Euclidean parallelism

1. Suppose that **L** and **M** are skew lines in $\mathbf{R}^3$. Prove that there is a unique plane **P** such that **L** is contained in **P** and **M** is parallel to (*i.e.*, disjoint from) **P**. [*Hints:* Write **L** and **M** as **x + V** and **y + W**, where **V** and **W** are $1-$dimensional vector subspaces. Since **L** and **M** are not parallel, we know that **V** and **W** are distinct. Let **U** be the vector subspace **V + W**. Why is **U** a $2-$dimensional subspace? Set **P** = **x + U** and verify that **P** has the desired properties; in particular, if **M** and **P** have a point **z** in common, note that **M** = **z + W** and **P** = **z + U.**]

2. Suppose that **P** and **Q** are parallel planes in $\mathbf{R}^3$, and let **S** be a plane which meets each of them in a line. Prove that the lines of intersection **S** ∩ **P** and **S** ∩ **Q** must be parallel.

3. Suppose that we are given three distinct planes **S, P** and **Q** in $\mathbf{R}^3$ such that **S** is parallel to both **P** and **Q.** Prove that **P** and **Q** are parallel. [*Hint:* Through a given point **x** not on **S,** how many planes are there that pass through **x** and are parallel to **S?**]

4. Suppose that **v** and **w** are linearly independent vectors in $\mathbf{R}^3$, and let **z** be a third vector in $\mathbf{R}^3$. Let a and b be real numbers. Prove that $f(a, b) = |z - av - bw|$ takes a minimum value when $z - av - bw$ is perpendicular to both **v** and **w**. [*Hint:* Take an orthonormal basis **e, f** for the span of **v** and **w,** and let $Q(z) = z - \langle z, e \rangle e - \langle z, f \rangle f$. Why is **z** perpendicular to **e, f, v,** and **w?** Find the length squared of the vector

$$z - s\,e - t\,f \;\; = \;\; Q(z) - \big[s - \langle z, e \rangle\big]e \; - \; \big[t - \langle z, f \rangle\big]f$$

using the fact that **e, f** and **Q(z)** are mutually perpendicular. Why is the (square of the) length minimized when the coefficients of **e** and **f** are zero? Why is the minimum value of $g(s, t) = |z - s\,e - t\,f|$ equal to the minimum value of $f(a, b) = |z - av - bw|$?]

5. Suppose that we have two skew lines in $\mathbf{R}^3$ of the form **0a** and **bc.** Let **x** and **y** be points of **0a** and **bc** respectively. Prove that the distance $d(x, y) = |x - y|$ is minimized when **x – y** is perpendicular to **a** and **c – b.** (In other words, ***the shortest distance between the two skew lines is along their common perpendicular.***)

6. Let **W, X, Y, Z** be four points in $\mathbf{R}^3$, no three of which are collinear, let **A, B, C, D** be the midpoints of **WX, XY, YZ,** and **ZW,** and suppose that we have **AB** ≠ **CD** and **AD** ≠ **BC.** Prove that **AB || CD** and **AD || BC.**

7. Let ∠**ABC** be given, and let **D** lie in the interior of ∠**ABC.** Prove that **D** lies on an open segment **(XY),** where **X** ∈ **(BA** and **Y** ∈ **(BC.** [*Hint:* The proof requires the use of Playfair's Axiom. In Unit **V** there is an exercise which shows that result is not necessarily true if Playfair's axiom does not hold.]