Exercises for Unit V
(Introduction to non – Euclidean geometry)

V.1 : Facts from spherical geometry

1. Two points on a circle Γ are said to be ***diametrically opposite*** if the center of the circle lies between them. Suppose that **A** and **B** are points on Γ that are not diametrically opposite, and let **M** and **m** be the major and minor arcs of Γ determined by **A** and **B**. Prove that **M** contains pairs of diametrically opposite points, but **m** does not contain any pairs of diametrically opposite points. **[*Hint:*** If the circle Γ has center **Q** and contains the points **A** and **B** such that **A** and **B** are not diametrically opposite each other, then the minor arc of the circle determined by **A** and **B** is the union of **{ A, B }** and the intersection of Γ with the interior of $\angle$**AQB**, and the corresponding major arc is the union of **{ A, B }** and the set of all points in Γ that do not lie on the minor arc.**]**

2. A basic theorem of geometry states that *the intersection of a sphere and a plane is either empty, a single point or a circle.* Find the radius of the circle of intersection for the sphere with equation $x^2 + y^2 + z^2 = 1$ and the plane with equation $x + y + z = 1$. **[*Hint:*** If the sphere Σ and plane **P** have an intersection which consists of more than one point, then the center **z** of the circle where they intersect is the foot of the perpendicular from **z** to **P**. In particular, if Σ is given by the equation $|\mathbf{x}|^2 = 1$ and **P** is given by an equation of the form $\mathbf{a} \cdot \mathbf{x} = b$ where **a** is nonzero, then the perpendicular line is the subspace spanned by the vector **a.**]

3. Suppose that we are given two spheres Σ_1 and Σ_2 which have different centers and at least one point in common. Prove that their intersection is contained in a plane Π which is perpendicular to the line joining their centers, and furthermore $\Sigma_1 \cap \Sigma_2 = \Sigma_1 \cap \Pi = \Sigma_2 \cap \Pi$. **[*Hint:*** Use vectors.**]**

4. Suppose that the sphere Σ and the plane **P** intersect in exactly one point **X,** and suppose that **Q** is the center of Σ. Prove that **QX** is perpendicular to **P.** Conversely, show that if the sphere Σ and the plane **P** intersect at **X** such that **QX** is perpendicular to **P,** then Σ and **P** have exactly one point in common. **[*Hint:*** These are analogs of the results for tangent lines to circles in plane geometry.**]**

V.2 : Attempts to prove Euclid's Fifth Postulate

1. For each of the following theorems in plane geometry, give a proof that does ***not*** use Playfair's Postulate (***such a proof must be synthetic*** and ***not analytic or based upon linear algebra***).

(a) Let **A** and **B** be distinct points, and let x be a positive real number. Then there is a unique point **Y** on the open ray **(AB** such that $d(\mathbf{A},\mathbf{Y}) = x$. Furthermore, we have **A*Y*B** if and only if $x < d(\mathbf{A},\mathbf{B})$, and likewise we have **A*B*Y** if and only if $x > d(\mathbf{A},\mathbf{B})$. **[*Hint:*** Use the Ruler Postulate.**]**

(b) If **L** is a line and **X** is a point not on **L**, then there is a unique line through **X** which is perpendicular to **L**. **[*Hint:*** Let **A** and **B** be points on **L**, and find a point **Y** on the side of **L** opposite **X** such that $\triangle\mathbf{XAB} \cong \triangle\mathbf{YAB}$. Show that **L** is the perpendicular bisector of **[XY].** Use the Exterior Angle Theorem to show there cannot be two perpendiculars from **X** with feet **C** and **D.]**

(c) The Classical Triangle Inequality. **[*Hint:*** Given $\triangle$**ABC,** let **D** be such that **A*B*D** and $d(\mathbf{B},\mathbf{D}) = d(\mathbf{B},\mathbf{C})$. Show that $|\angle\mathbf{ADC}| < |\angle\mathbf{ACD}|$, and explain why the conclusion follows from this.**]**

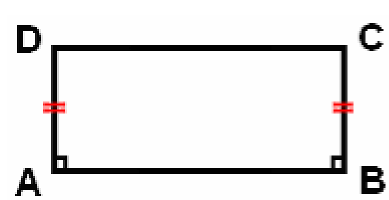
(d) The **AAS** Triangle Congruence Theorem. **[*Hint:*** Given $\triangle$**ABC** and $\triangle$**DEF** with $d(\mathbf{A},\mathbf{B}) = d(\mathbf{D},\mathbf{E})$, $|\angle\mathbf{ABC}| = |\angle\mathbf{DEF}|$ and $|\angle\mathbf{ACB}| = |\angle\mathbf{DFE}|$, we know that $\triangle\mathbf{ABC} \cong \triangle\mathbf{DEF}$ if $d(\mathbf{A},\mathbf{C}) = d(\mathbf{D},\mathbf{F})$, so assume this is false. Explain why, without loss of generality, we may assume $d(\mathbf{A},\mathbf{C}) > d(\mathbf{D},\mathbf{F})$. Let **G** be a point of **(AC)** such that that $d(\mathbf{A},\mathbf{G}) = d(\mathbf{D},\mathbf{F})$, so that $\triangle\mathbf{ABG} \cong \triangle\mathbf{DEF}$. Why does this imply that $|\angle\mathbf{AGB}| = |\angle\mathbf{ACB}|$, and why does this contradict the Exterior Angle Theorem**?]**

V.3 : Neutral geometry

1. Suppose that p and q are arbitrary positive real numbers. Prove that there is a Saccheri quadrilateral □**ABCD** with base **AB** such that $d(\mathbf{A},\mathbf{D}) = d(\mathbf{B},\mathbf{C}) = p$ and $d(\mathbf{A},\mathbf{B}) = q$.

2. Prove the following consequence of the Archimedean Law that was stated in the notes: *If h and k are positive real numbers, then there is a positive integer n such that* $h/2^n < k$. **[*Hint:*** Why is there a positive integer n such that $1/n < k/h$? Use this and the inequality $n < 2^n$.**]**

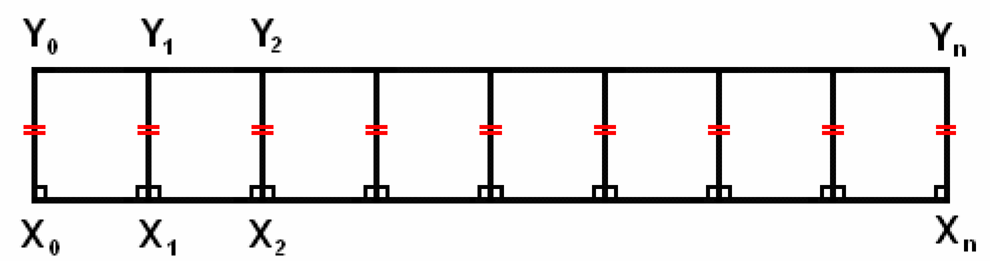
Standing hypotheses: In Exercises 2 – 5 below, points **A, B, C, D** in a neutral plane form the vertices of a Saccheri quadrilateral such that **AB** is perpendicular to **AD** and **BC,** and $d(\mathbf{A},\mathbf{D}) = d(\mathbf{B},\mathbf{C})$. The segment **[AB]** is called the ***base,*** the segment **[CD]** is called the ***summit,*** and **[AD]** and **[BC]** are called the ***lateral sides.*** The vertex angles at **C** and **D** are called the ***summit angles.***



3. Prove that the summit angles at **C** and **D** have equal measures. [***Hint:*** Why do the diagonals have equal length? Use this fact to show that $\triangle$**BDC** $\cong$ $\triangle$**ACD.**]

4. Prove that the line joining the midpoints of **[AB]** and **[CD]** is perpendicular to both **[AB]** and **[CD].** [***Hint:*** Imitate a similar argument from an earlier set of exercises for isosceles trapezoids in Euclidean geometry.]

5. Prove that $d(\mathbf{A}, \mathbf{B}) \leq d(\mathbf{C}, \mathbf{D})$. [***Hint:*** First show by induction that if $\mathbf{G}_1, \dots, \mathbf{G}_n$ are arbitrary points, then $d(\mathbf{G}_1, \mathbf{G}_n) \leq d(\mathbf{G}_1, \mathbf{G}_2) + \dots + d(\mathbf{G}_{n-1}, \mathbf{G}_n)$. Next, show that one can find Saccheri quadrilaterals as in the picture below such that Saccheri quadrilateral **ABCD** is identical to Saccheri quadrilateral $\mathbf{X}_0\mathbf{X}_1\mathbf{Y}_1\mathbf{Y}_0$ in the picture, with all summits and all bases having equal lengths. The idea of the construction should be clear from the picture, but some work is needed to show that each $d(\mathbf{Y}_j, \mathbf{Y}_{j+1})$ is equal to $d(\mathbf{C}, \mathbf{D})$; it is helpful to start by considering the diagonals $[\mathbf{Y}_j\, \mathbf{X}_{j+1}]$ and the two triangles into which they split the Saccheri quadrilateral $\mathbf{X}_j\mathbf{X}_{j+1}\mathbf{Y}_{j+1}\mathbf{Y}_j$.



Using these observations, show that we have

$$n\, d(\mathbf{A}, \mathbf{B}) = d(\mathbf{X}_0, \mathbf{X}_n) \leq$$

$$d(\mathbf{X}_0, \mathbf{Y}_0) + d(\mathbf{Y}_0, \mathbf{Y}_1) + \dots + d(\mathbf{Y}_{n-1}, \mathbf{Y}_n) + d(\mathbf{Y}_n, \mathbf{X}_n) = n\, d(\mathbf{C}, \mathbf{D}) + 2d(\mathbf{A}, \mathbf{D}).$$

This means that $d(\mathbf{A}, \mathbf{B}) \leq d(\mathbf{C}, \mathbf{D}) + 2d(\mathbf{A}, \mathbf{D})/n$ for every positive integer n. Why can the second summand be made smaller than any positive real number, and why does this imply $d(\mathbf{A}, \mathbf{B}) \leq d(\mathbf{C}, \mathbf{D})$? Alternatively, why does this imply that $d(\mathbf{A}, \mathbf{B}) > d(\mathbf{C}, \mathbf{D})$ must be false?]

6. Suppose we are given Saccheri quadrilaterals $\square$**ABCD** and $\square$**EFGH** with right angles at **A, B** and **E, F** such that the lengths of the bases and lateral sides in $\square$**ABCD** and $\square$**EFGH** are equal. Prove that the lengths of the summits and the measures of the summit angles in $\square$**ABCD** and $\square$**EFGH** are equal.

7. Suppose are given Lambert quadrilaterals $\square$**ABCD** and $\square$**EFGH** with right angles at **A, B, C** and **E, F, G** such that $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{E}, \mathbf{F})$ or $d(\mathbf{B}, \mathbf{C}) = d(\mathbf{F}, \mathbf{G})$. Prove that $d(\mathbf{C}, \mathbf{D}) = d(\mathbf{G}, \mathbf{H})$, $d(\mathbf{A}, \mathbf{D}) = d(\mathbf{E}, \mathbf{H})$, and $|\angle \mathbf{CDA}| = |\angle \mathbf{GHE}|$.

8. In the setting of Exercise 5, prove that if we have $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{C}, \mathbf{D})$ then the Saccheri quadrilateral $\square$**ABCD** is a rectangle.

9. Suppose that the points **A, B, C, D** in a neutral plane form the vertices of a Lambert quadrilateral with right angles at **A, B, C.** Prove that $d(\mathbf{A}, \mathbf{D}) \leq d(\mathbf{B}, \mathbf{C})$ and $d(\mathbf{A}, \mathbf{B}) \leq d(\mathbf{C}, \mathbf{D})$. [***Hint:*** Start by explaining why it suffices to prove the first of these.

Show that there is a Saccheri quadrilateral □ **AEFD** such that **B** and **C** are the midpoints of the base **[AE]** and the summit **[FD]** respectively. Apply Exercise 4.]

10. In the setting of the preceding exercise, prove that if we have $d(\mathbf{A}, \mathbf{B}) = d(\mathbf{C}, \mathbf{D})$ or $d(\mathbf{A}, \mathbf{D}) = d(\mathbf{B}, \mathbf{C})$, then the Lambert quadrilateral □ **ABCD** is a rectangle.

11. Suppose that $\mathscr{P}$ is a neutral plane, and let p and q be arbitrary positive real numbers. Prove that there is a Lambert quadrilateral □**ABCD** with right angles at **A, B,** and **C** such that $d(\mathbf{A}, \mathbf{D}) = p$ and $d(\mathbf{A}, \mathbf{B}) = q$. [***Hint:*** One can view a Lambert quadrilateral as half of a Saccheri quadrilateral.]

12. Suppose that $\mathscr{P}$ is a neutral plane, and let p and q are arbitrary positive real numbers for which that there is a Lambert quadrilateral □**ABCD** with right angles at **A, B,** and **C** such that $d(\mathbf{A}, \mathbf{B}) = p$ and $d(\mathbf{B}, \mathbf{C}) = q$. Let s be a positive real number such that $s < p$. Prove that there is a Lambert quadrilateral □**WXYZ** with right angles at **W, X,** and **Y** such that $d(\mathbf{W}, \mathbf{X}) = s$ and $d(\mathbf{X}, \mathbf{Y}) = q$. [***Hint:*** Let **E** be a point on **(BA)** such that $d(\mathbf{B}, \mathbf{E}) = s$. Take the perpendicular line to **AB** in $\mathscr{P}$ containing **E,** and explain why it must meet the diagonal segment **(BD)** and the opposite side **(CD)** using Pasch's Theorem.]

Note. For arbitrary positive real numbers p and q it is ***not always possible*** to find a Lambert quadrilateral □**ABCD** with right angles at **A, B,** and **C** such that $d(\mathbf{A}, \mathbf{B}) = p$ and $d(\mathbf{B}, \mathbf{C}) = q$. However, the proof involves a detailed analysis of the Poincaré model for hyperbolic geometry discussed in Sections V.5 – V.7 of the notes, and since our discussion of the model is only informal we cannot give a complete proof of this fact here. However, a drawing which depicts one such example (with $p = q$ sufficiently large) appears in the discussion of this exercise in the file

http://math.ucr.edu/~res/math133/solutions7figures.pdf

which can be found in the course directory.

13. Suppose that we are given △**ABC** and △**A′B′C′** in a neutral plane $\mathscr{P}$ such that △**ABC** ≅ △**A′B′C′**. Let **D, E, F** be the respective midpoints of **[BC], [AC]** and **[AB],** and let **D′, E′, F′** be the respective midpoints of **[B′C′], [A′C′]** and **[A′B′].**

- **(a)** Prove that △**AEF** ≅ △**A′E′F′,** and explain why we must also have △**BDF** ≅ △**B′D′F′** and △**CDE** ≅ △**C′D′E′.**
- **(b)** Prove that △**DEF** ≅ △**D′E′F′.**

14. In Euclidean geometry one can improve the preceding result to say that all eight triangles in **(a)** and **(b)** are congruent to each other (with the vertices suitably ordered). Write out the conclusion explicitly, and explain why it is true.

V.4 : Angle defects and related phenomena

1. Given a Saccheri quadrilateral in a hyperbolic plane $\mathcal{P}$, explain why the summit is always shorter than the base. If ☐**ABCD** is a Lambert quadrilateral in a hyperbolic plane $\mathcal{P}$ with right angles at **A, B,** and **C,** what can one say about the lengths of the pairs of opposite sides { **[AB], [CD]** } and { **[BC], [AD]** }? Give reasons for your answer.

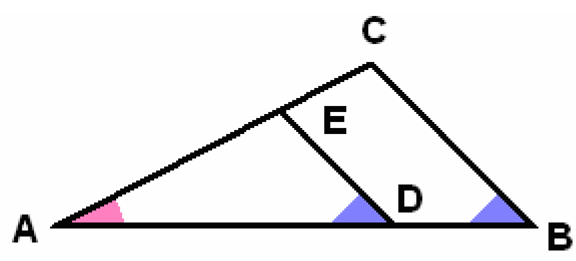
2. Given a Saccheri quadrilateral in a hyperbolic plane $\mathcal{P}$, show that the line segment joining the midpoints of the summit and base is shorter than the lengths of the lateral sides.

3. Suppose that ε is an arbitrary positive real number and $\mathcal{P}$ is a hyperbolic plane. Prove that there is a triangle in $\mathcal{P}$ whose angle defect is less than ε. [***Hint:*** Let △**ABC** have defect δ. If one splits it into two triangles, why will ***at least one*** of them have defect at most $\frac{1}{2}\delta$? Show that if one iterates this enough times, one obtains the desired triangle.]

4. Suppose we are given an isosceles triangle △**ABC** in a hyperbolic plane $\mathcal{P}$ with d(**A, B**) = d(**A, C**), and let **D** and **E** be points on **(AB)** and **(AC)** such that d(**A, D**) = d(**A, E**). Prove that |∠**ABC**| < |∠**ADE**|. [***Hint:*** Compare the angle defects of the two isosceles triangles in the problem.]

5. Suppose we are given an equilateral triangle △**ABC** in a hyperbolic plane $\mathcal{P}$, and let **D, E** and **F** be the midpoints of **[BC], [AC]** and **[AB].** Prove that △**DEF** is also an equilateral triangle and that |∠**ABC**| < |∠**DEF**|. Using the conclusion of the preceding exercise, explain why △**AEF** is ***not*** an equilateral triangle.

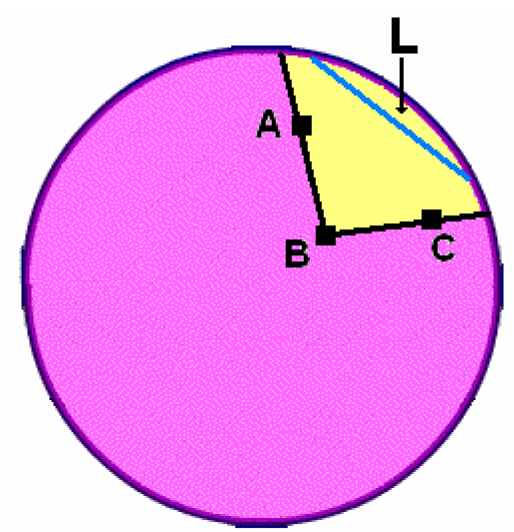
6. Prove the following result, which shows that Corollary **III.2.15** (the "Third angles are equal" property) fails completely in hyperbolic geometry: *Given a triangle* △**ABC** *in a hyperbolic plane* $\mathcal{P}$ *and a point* **D** *on* **(AB)**, *then there exists a point* **E** *on* **(AC)** *such that* |∠**ABC**| = |∠**ADE**| *but also* |∠**ACB**| < |∠**AED**|.



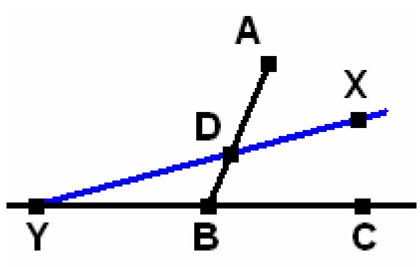
7. Suppose we are given a Saccheri quadrilateral ☐**ABCD** in a hyperbolic plane $\mathcal{P}$ with base **[AB],** and assume that the lengths of the base and lateral sides are equal. Does the ray **[AC** bisect ∠**DAB?** Give reasons for your answer.

8. Suppose we are given a hyperbolic plane $\mathcal{P}$. Prove that there is a line **L** and an angle $\angle$**ABC** in $\mathcal{P}$ such that **L** is contained in the interior of $\angle$**ABC**. **[*Hint:*** Let **B** be a point not on **L** such that there are at least two parallel lines to **L** through **B.** If **Y** is the foot of the perpendicular from **B** to **L** and **M** is a line through **B** which is perpendicular to **YB,** then **M** is parallel to **L,** and there is also a second line **N** which is parallel to **L.** Explain why there is a ray **[BA** on **N** such that **A** lies on the same side of **M** as **Y,** and explain why there is also a ray **[BC** on **M** such that **C** and **A** lie on opposite sides of **BY.** Why is **L** disjoint from $\angle$**ABC,** why does the point **Y** on **L** lie in the interior of this angle, and why does this imply that the entire line is contained in the interior of $\angle$**ABC?]**

Note. In fact, it turns out that ***every* angle $\angle$ABC in a hyperbolic plane $\mathcal{P}$ contains a line (in fact, *infinitely many* lines) in its interior.** One way of seeing this is to use the Beltrami – Klein model for hyperbolic geometry described later in Section **V.6.** An illustration is given below; as noted in Section **V.6,** the points in the model are the points inside the boundary circle, and the lines in the model are open chords whose endpoints lie on the boundary circle.



In this picture, $\angle$**ABC** is given in the Beltrami – Klein model of the hyperbolic plane (the interior of the disk, with the boundary excluded), and **L** is just one example (among infinitely many) of a line which is contained entirely in the interior of $\angle$**ABC.** — This contrasts sharply with Euclidean geometry, where ***a line containing a point of the interior of $\angle$ABC must meet the angle in at least one point.*** [***Proof:*** Suppose that **L** contains an interior point **X** of the angle; if **A** lies on **L** the result is true, so it suffices to consider the case where **A** does not lie on **L.** By Playfair's Axiom we know that at least one of the lines **BA** and **BC** must have a point in common with **L.** Interchanging the roles of **A** and **C** if necessary, we may assume that **BC** meets **L** at some point. If this common point **Y** lies on **[BC** we are done, so suppose it does not; in this case we know that **B** is between **Y** and **C.** The latter implies that **Y** and **C** lie on opposite sides of **AB;** since **X** and **C** are assumed to lie on the same side of **BA,** it follows that the open segment **(XY)** meets **AB** at some point **D.** Since **D** is between **X** and **Y** and **Y** lies on **BC,** it follows that **X** and **D** lie on the same side of **BC,** and since **A** and **X** also lie on the same side of **AC** (by the assumption on **X**) it follows that the rays **[BA** and **[BD** are identical. But this means that **L** and **[BA** have a point in common; thus we see that in all cases the line **L** has at least one point in common with $\angle$**ABC.** — The drawing below may be helpful for understanding the argument we have described in this paragraph.■]



V.5 : Further topics in hyperbolic geometry

No exercises.

V.6 : Subsequent developments

No exercises.

V.7 : Non – Euclidean geometry in modern mathematics

The following exercises are related to the geometric properties of the Poincaré disk model for the hyperbolic plane.

1. Let **a** be a vector in $\mathbb{R}^2$, let $r > 0$, and let $\Gamma(r;\mathbf{a})$ denote the circle in $\mathbb{R}^2$ with center **a** and radius r. The transformation

$$\mathbf{T}(\mathbf{v}) \;=\; \mathbf{a} + \frac{r^2}{|\mathbf{v}-\mathbf{a}|^2}\cdot(\mathbf{v}-\mathbf{a})$$

is called ***inversion with respect to the circle*** $\Gamma(r;\mathbf{a})$. It is defined for all $\mathbf{v} \neq \mathbf{a}$, and its image is the same set. Prove that **T** is $1-1$ onto, and in fact for all $\mathbf{v} \neq \mathbf{a}$ the point $\mathbf{T}(\mathbf{v})$ lies on the ray through **v** with origin **a** such that $|\mathbf{T}(\mathbf{v})-\mathbf{a}|\cdot|\mathbf{v}-\mathbf{a}| = r^2$; using this description, explain why **T** is equal to its own inverse.

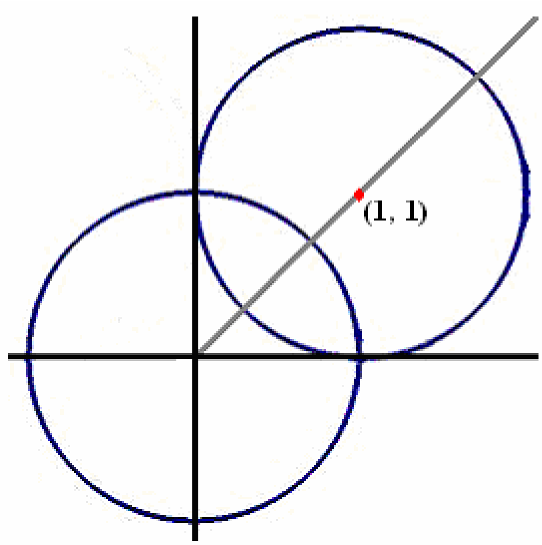
Note. Two motivations for the name "inversion" are that **T** is its own inverse and if we view points in $\mathbb{R}^2$ as complex numbers, then inversion with respect to the standard unit circle $\Gamma(1;0)$ sends a complex number z to the ***conjugate*** of z^{-1}. A particularly simple formula for the specific inversion mapping **T** is given by $\mathbf{T}(\mathbf{v}) = |\mathbf{v}|^{-2}\mathbf{v}$.

2. Let **T** and **S** be inversions with respect to the circle $\Gamma(1;0)$ and $\Gamma(r;0)$, where $r > 0$. Prove that $\mathbf{S}(\mathbf{v}) = r\,\mathbf{T}(r^{-1}\mathbf{v})$ for all nonzero vectors **v**.

3. Let **T** be inversion with respect to the standard unit circle $\Gamma(1;0)$, let Γ_1 be a circle which does not contain the origin, and let Γ_2 be the set of all points of the form **T(v)** for some point **v** on Γ_1. Prove that Γ_2 is also a circle. [***Hints*:** There are two cases, depending upon whether or not the origin is the center of Γ_1. If the center of the latter is the origin and the radius is k, explain why Γ_2 is the circle whose center is the origin and whose radius is $1/k$. On the other hand, if the center **a** of Γ_1 is not the origin, then explain why $|\mathbf{a}| \neq k$ and therefore $q = |\mathbf{a}|^2 - k^2$ is nonzero. Show that the dot product $2\mathbf{a}\cdot\mathbf{v} = |\mathbf{v}|^2 + q$ and if **v** lies on Γ_1, then the distance between **T(v)** and the vector $q^{-1}\mathbf{a}$ is equal to $|q|^{-1}k$. This shows that Γ_2 is contained in a circle; to prove that every point of this circle is the image of a point in Γ_1, show that **T** also maps the circle described above into Γ_1.]

4. Let **T** be inversion with respect to the standard unit circle $\Gamma(1;0)$, and let Γ_1 be the circle defined by the equation $x^2 + (y-b)^2 = b^2$, where b is positive. Show that the image of the nonzero points in Γ_1 under **T** is the horizontal line defined by the equation $y = 1/(2b)$. As in the preceding exercise, there are two parts: The first step is to show that **T** maps the nonzero points of Γ_1 into the horizontal line, and the second is to show that **T** maps the points of the given horizontal line into Γ_1.

5. Let **T** be inversion with respect to the standard unit circle $\Gamma(1;0)$, and let Γ_1 be the circle defined by the equation $(x-1)^2 + (y-1)^2 = 1$. Prove that the circles $\Gamma(1;0)$ and Γ_1 meet in two points and that their tangent lines are perpendicular at these two points. Also, prove that if **v** is a point of the circle Γ_1 then so is its image **T(v)**. [***Hints*:** There is a unique circle which passes through three noncollinear points, so if **T** sends three points of Γ_1 into Γ_1 it must send all points of Γ_1 into Γ_1. It may also be very helpful to graph the two circles before trying to solve this exercise; a sketch is given below.]



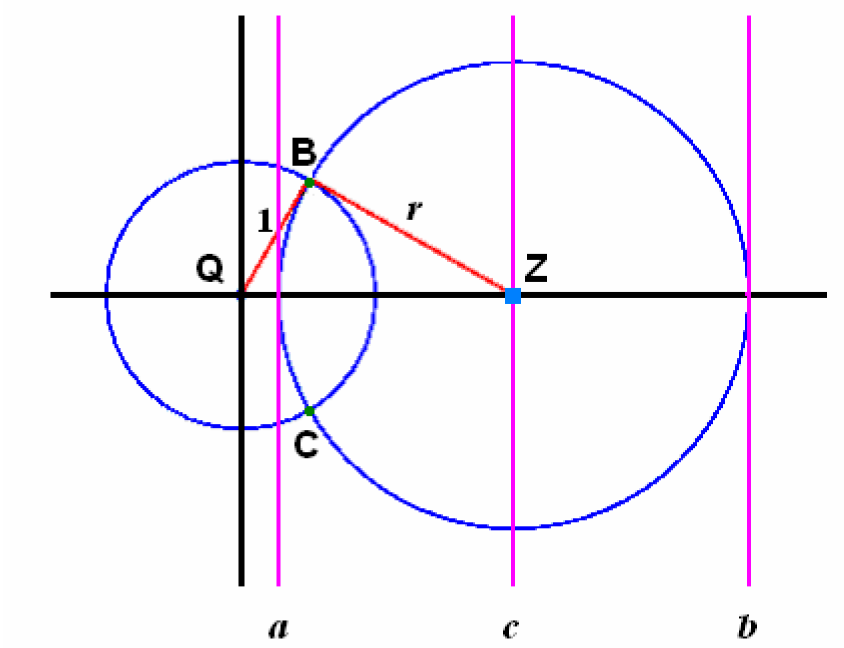
6. Let $\Gamma_0 = \Gamma(1;0)$, let **Q** = $(0, 0)$, and let Γ_1 be a circle of radius r with center **Z** = $(c, 0)$, where $c > 0$, such that Γ_1 meets the x – axis at points $(a, 0)$ and $(b, 0)$, where $0 < a < 1 < b$; under these conditions the Two – Circle Theorem implies that Γ_0 and Γ_1 meet at two points, say **B** and **C.**

(i) Prove that the two circles meet orthogonally (in other words, **QB** is perpendicular to **ZB** and **QC** is perpendicular to **ZC**) if and only if $b = 1/a$. **[*Hints*:** One can solve for r and c in terms of a and b because **Z** is the midpoint of the segment joining $(a, 0)$ to $(b, 0)$ and $2r = b - a$. Apply the Pythagorean Theorem and its converse.**]**

(ii) Using the first part of this exercise, prove that if **Y** is an arbitrary point in the interior of Γ_0 such that **Y** is not the circle's center, then there is a circle Γ_1 which contains **Y** and meets Γ_0 orthogonally.

(iii) Suppose that Γ_0 and Γ_1 satisfy the conditions in the first part of this exercise. Let **T** be inversion with respect to Γ_0. Prove that **v** lies on Γ_1 if and only if **T(v)** does.

Here is a drawing to illustrate the objects in this exercise.



By the Pythagorean Theorem and its converse, there is a right angle at **B** if and only if we have $r^2 + 1 = c^2$.