SOLUTIONS TO EXERCISES FOR

MATHEMATICS 133 — Part 1

Winter 2009

I. Topics from linear algebra

I.0 : Background

1. Suppose that $\{\mathbf{x}, \mathbf{y}\}$ is linearly dependent. Then there are scalars a, b which are not both zero such that $a\mathbf{x} + b\,\mathbf{y} = \mathbf{0}$. — In fact, we claim that both a and b must be nonzero. If $a = 0$ then the equation reduces to $b\,\mathbf{y} = \mathbf{0}$, where both factors on the right hand side are nonzero. Since a product of a nonzero scalar and a nonzero vector is never zero, this is impossible. This contradiction arises from our supposition that a was zero, and hence we see that $a \neq 0$. Similar considerations show that b must also be nonzero. Since we now know that a and b are both nonzero, it follows that

$$\mathbf{y} = b^{-1}a\mathbf{x} \qquad \text{and} \qquad \mathbf{x} = a^{-1}b\,\mathbf{y}$$

and hence $\mathbf{x}$ and $\mathbf{y}$ are nonzero multiples of each other.

Conversely, suppose we know that $\mathbf{x} = p\,\mathbf{y}$ or $\mathbf{y} = q\mathbf{x}$ where (respectively) p or q is nonzero. Then we have (resepectively) $\mathbf{x} - p\,\mathbf{y} = \mathbf{0}$ and $q\mathbf{x} - \mathbf{y} = \mathbf{0}$, which shows that the set $\{\mathbf{x}, \mathbf{y}\}$ is linearly dependent.$\blacksquare$

2. Assume the conclusion is false, so that $S \cup \{\mathbf{z}\}$ is linearly dependent. Then there exist scalars $a_1, \cdots, a_n, b$ which are not all zero such that

$$a_1\mathbf{v}_1 + \ \cdots\ + a_n\mathbf{v}_n + b\,\mathbf{z} \quad = \quad \mathbf{0}\ .$$

We claim that $b \neq 0$; if on the contrary we have $b = 0$, then at least one of the remaining scalars a_j must be nonzero and it will follow that $\{\,\mathbf{v}_1,\ \cdots\ ,\mathbf{v}_n\,\}$ must be linearly dependent, contradicting our hypothesis.

Sine $b \neq 0$ we can solve for $\mathbf{z}$ to obtain

$$\mathbf{z} \quad = \quad -\big(\,b^{-1}a_1\mathbf{v}_1 + \ \cdots\ + b^{-1}a_n\mathbf{v}_n\,\big)$$

which contradicts our hypothesis that $\mathbf{z}$ is not a linear combination of the vectors in $\{\,\mathbf{v}_1,\ \cdots\ ,\mathbf{v}_n\,\}$. The source of this contradiction is the assumption that the linear independence conclusion is false, and therefore it follows that the set $S \cup \{\mathbf{z}\}$ must be linearly independent.$\blacksquare$

3. We shall show that the original set S is linearly **dependent** if and only if the second set $T = \{\,c_1\mathbf{v}_1,\ \cdots\ ,c_n\mathbf{v}_n\,\}$ is linearly dependent. If S is dependent then there exist scalars $a_1, \cdots, a_n$ which are not all zero such that

$$a_1\mathbf{v}_1 + \ \cdots\ + a_n\mathbf{v}_n \quad = \quad \mathbf{0}\ .$$

Clearly we may rewrite this in the form

$$(a_1c_1^{-1})c_1\mathbf{v}_1 + \ \cdots\ + (a_nc_n^{-1})c_n\mathbf{v}_n \quad = \quad \mathbf{0}$$

and since the scalars c_j are not all zero it follows that the scalars $a_1 c_1^{-1}, \cdots, a_n c_n^{-1}$ are also not all equal to zero. Therefore the set T is linearly dependent. — Conversely, if T is linearly dependent, then there exist scalars $b_1, \cdots, b_n$ which are not all zero such that

$$b_1 c_1 \mathbf{v}_1 + \ \cdots \ + b_n c_1 \mathbf{v}_n \quad = \quad \mathbf{0} \ .$$

Since the numbers c_j are not all zero, it follows that the numbers $a_j = b_j c_j$ are not all zero and that

$$a_1 \mathbf{v}_1 + \ \cdots \ + a_n \mathbf{v}_n \quad = \quad \mathbf{0}$$

which means that S must be linearly dependent.■

4. We shall show that if S is linearly dependent then so is $T[S]$, and if $T[S]$ is linearly dependent then so is S,

Suppose first that T is linearly dependent, so that there exist scalars $a_1, \cdots, a_n$ which are not all zero such that

$$a_1 \mathbf{v}_1 + \ \cdots \ + a_n \mathbf{v}_n \quad = \quad \mathbf{0} \ .$$

By the linearity of T it follows that

$$a_1 T(\mathbf{v}_1) + \ \cdots \ + a_n T(\mathbf{v}_n) \quad = \quad \mathbf{0}$$

and hence $T[S]$ is linearly dependent.

Now suppose that S is linearly independent. Suppose we have scalars a_j for which

$$a_1 T(\mathbf{v}_1) + \ \cdots \ + a_n T(\mathbf{v}_n) \quad = \quad \mathbf{0} \ .$$

Now the linearity of T implies the left hand side is equal to

$$T\big(\, a_1 \mathbf{v}_1 + \ \cdots \ + a_n \mathbf{v}_n \,\big)$$

and hence T sends the vector $a_1 \mathbf{v}_1 + \ \cdots \ + a_n \mathbf{v}_n$ to zero. Since T is 1–1 and T sens the zero vector to the zero vector, it follows that

$$a_1 \mathbf{v}_1 + \ \cdots \ + a_n \mathbf{v}_n \quad = \quad \mathbf{0} \ .$$

The linear independence of S now implies that $a_j = 0$ for all j, and therefore we see that $T[S]$ satisfies the defining condition for a linearly independent set.■

I.1 : Dot products

1. The answer is -6.■

2. The answer is 2.■

3. The answer is 1.■

4. The two vectors are not perpendicular because their dot product is equal to 4, but they are linearly independent because neither is a scalar multiple of each other (and both are nonzero); the latter is apparent because every scalar multiple of the first vector has a zero in the second coordinate and this is not true for the second vector.■

5. They are perpendicular because their dot product is zero. Both vectors are nonzero so they are linearly independent.■

6. The two vectors are perpendicular because their dot product is equal to $-2+3-1=0$, and since they are both nonzero they are also linearly independent.■

7. Let $\mathbf{x} = (k,k,0)$ and $\mathbf{y} = (0,k,k)$, so that we have the following:

$$\mathbf{x}-\mathbf{z} \;=\; \tfrac{1}{2}(k,k,-k) \qquad\qquad \mathbf{y}-\mathbf{z} \;=\; \tfrac{1}{2}(-k,k,k)$$

$$|\,\mathbf{x}-\mathbf{z}\,| \;=\; |\,\mathbf{y}-\mathbf{z}\,| \;=\; \frac{k\sqrt{3}}{2} \qquad\qquad (\,\mathbf{x}-\mathbf{z}\,)\cdot(\,\mathbf{y}-\mathbf{z}\,) \;=\; \frac{-k^2}{4}$$

If we substitute this into the formula for $\cos \angle\mathbf{xzy}$ we find that the latter is equal to $-\frac{1}{3}$, so that $|\,\angle\mathbf{xzy}\,| = \mathrm{Arc}\,\cos(-\frac{1}{3}) \approx 109.47122°$.■

8. This and the following exercises are based upon the following identities:

$$\mathbf{u}_0 \;=\; \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\cdot\mathbf{v}\;, \qquad\qquad \mathbf{u}_1 \;=\; \mathbf{u}-\mathbf{u}_0$$

For the examples in this problem we have $\mathbf{u}\cdot\mathbf{v} = 13$ and $\mathbf{v}\cdot\mathbf{v} = 26$, so that $\mathbf{u}_0 = \frac{1}{2}\mathbf{v} = (\frac{5}{2},\frac{1}{2})$ and $\mathbf{u}_1 = \mathbf{u}-\mathbf{u}_0 = (-\frac{1}{2},\frac{5}{2})$.■

9. For the examples in this problem we have $\mathbf{u}\cdot\mathbf{v} = 11$ and $\mathbf{v}\cdot\mathbf{v} = 25$, so that $\mathbf{u}_0 = \frac{11}{25}\mathbf{v} = (0,\frac{33}{25},\frac{44}{25})$ and $\mathbf{u}_1 = \mathbf{u}-\mathbf{u}_0 = (2,-\frac{8}{25},\frac{6}{25})$. **Note.** In problems like this it is always a good idea to check the correctness of the calculuations by computing the dot product of $\mathbf{u}_1$ with $\mathbf{v}$ explicitly and checking that it is equal to zero.■

10. For the examples in this problem we have $\mathbf{u}\cdot\mathbf{v} = 21$ and $\mathbf{v}\cdot\mathbf{v} = 26$, so that $\mathbf{u}_0 = \frac{21}{26}\mathbf{v} = (-\frac{21}{26},\frac{78}{26},\frac{104}{26})$ and $\mathbf{u}_1 = \mathbf{u}-\mathbf{u}_0 = \frac{1}{26}(151,93,-32)$. — Given the relatively complex nature of the answer, it is particularly important in this case to check the perpendicularity condition as in the preceding exercise.■

11. For the examples in this problem we have $\mathbf{u}\cdot\mathbf{v} = -4$ and $\mathbf{v}\cdot\mathbf{v} = 14$, so that $\mathbf{u}_0 = -\frac{2}{7}\mathbf{v} = (-\frac{4}{7},-\frac{2}{7},\frac{6}{7})$ and $\mathbf{u}_1 = \mathbf{u}-\mathbf{u}_0 = (-\frac{3}{7},\frac{9}{7},\frac{1}{7})$.

12. We shall take these in order.

(a) If we expand the expression given in this part of the problem we obtain

$$\mathbf{u}\cdot\mathbf{v} \;+\; \mathbf{u}\cdot\mathbf{w} \;+\; \mathbf{v}\cdot\mathbf{v} \;+\; \mathbf{v}\cdot\mathbf{w} \;\;=$$

$$2 \;+\; 5 \;+\; 4 \;+\; 49 \;\;=\;\; 60\;.■$$

(b) If we expand the expression given in this part of the problem we obtain

$$6(\mathbf{v}\cdot\mathbf{u}) \;-\; 3(\mathbf{w}\cdot\mathbf{u}) \;+\; 4(\mathbf{v}\cdot\mathbf{w}) \;-\; 2(\mathbf{w}\cdot\mathbf{w}) \;\;=$$

$$(6\cdot(-3)\,) \;-\; (3\cdot 5) \;+\; (4\cdot(-3)\,) \;-\; (2\cdot 49) \;\;=\;\; -18 \;-\; 15 \;-\; 12 \;-\; 98 \;\;=\;\; -143\;.■$$

(c) If we expand the expression given in this part of the problem we obtain

$$4(\mathbf{u}\cdot\mathbf{u}) \;-\; 4(\mathbf{u}\cdot\mathbf{v}) \;+\; 8(\mathbf{w}\cdot\mathbf{u}) \;+\; (\mathbf{u}\cdot\mathbf{v}) \;-\; (\mathbf{v}\cdot\mathbf{v}) \;-\; 2(\mathbf{w}\cdot\mathbf{v}) \;\;=$$

$$(4 \cdot 1) \; - \; (4 \cdot 2) \; + \; (8 \cdot 5) \; + \; 2 \; - \; 4 \; - \; (2 \cdot (-3)\,) \quad =$$

$$4 \; - \; 8 \; + \; 40 \; - \; 2 \; + \; 6 \quad = \quad 40 \;. \blacksquare$$

(d) If we expand the square of the expression given in this part of the problem we obtain

$$(\mathbf{u} \cdot \mathbf{u}) \; + \; 2(\mathbf{u} \cdot \mathbf{v}) \; + \; (\mathbf{v} \cdot \mathbf{v}) \quad =$$

$$1 \; + \; (2 \cdot 2) \; + \; 4 \quad = \quad 9 \;. \blacksquare$$

Thus the value of the original expression is 9.

(e) If we expand the square of the expression given in this part of the problem we obtain

$$4(\mathbf{w} \cdot \mathbf{w}) \; 4(\mathbf{w} \cdot \mathbf{v}) \; + \; (\mathbf{v} \cdot \mathbf{v}) \quad =$$

$$(4 \cdot 49) \; - \; (4 \cdot (-3)\,) \; + \; 4 \quad = \quad 196 \; + \; 12 \; + \; 4 \quad = \quad 212 \;. \blacksquare$$

(f) If we expand the square of the expression given in this part of the problem we obtain

$$(\mathbf{u} \cdot \mathbf{u}) \; - \; 4(\mathbf{u} \cdot \mathbf{v}) \; - \; 8(\mathbf{u} \cdot \mathbf{w}) \; - \; 16(\mathbf{v} \cdot \mathbf{w}) \; + \; 4(\mathbf{v} \cdot \mathbf{v}) \; + \; 16(\mathbf{w} \cdot \mathbf{w}) \quad =$$

$$1 \; - \; (4 \cdot 2) \; - \; (8 \cdot 5) \; - \; (16 \cdot (-3)\,) \; + \; (4 \cdot 4) \; + \; (16 \cdot 49) \quad = \quad 8001 \;. \blacksquare$$

13. Again, we take each part separately:

(a) The orthonormal basis is given by the rows of the following matrix:

$$\frac{1}{\sqrt{6}} \cdot \begin{pmatrix} \sqrt{3} & \sqrt{3} & 0 \\ -1 & 1 & 2 \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} \end{pmatrix} \;.$$

(b) The orthonormal basis is given by the rows of the following matrix:

$$\frac{1}{\sqrt{6}} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2\sqrt{3} & 0 & 2\sqrt{3} \\ 0 & 1 & 2 & -1 \\ 0 & -\sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix} \;.$$

(c) The orthonormal basis is given by the rows of the following matrix:

$$\frac{1}{\sqrt{6}} \cdot \begin{pmatrix} 1 & 2 & 1 \\ -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\ -\sqrt{3} & 0 & \sqrt{3} \end{pmatrix} \;.$$

14. By the Pythagorean Formula, if $\mathbf{x}$ and $\mathbf{y}$ are perpendicular vectors we have

$$|\mathbf{x} + \mathbf{y}|^2 \quad = \quad |\mathbf{x}|^2 \; + \; |\mathbf{y}|^2$$

and if we use this repeatly on a set of mutually perpendicular vectors $\mathbf{x}_1, \; \cdots \; , \mathbf{x}_n$ we obtain the formula

$$|\mathbf{x}_1 \; + \; \cdots \; + \; \mathbf{x}_n|^2 \quad = \quad |\mathbf{x}_1|^2 \; + \; \cdots \; + \; |\mathbf{x}_n|^2 \;.$$

We know that $\mathbf{w} = \sum_j c_j \mathbf{v}_i$ for suitable coefficients c_j, and if we take inner products of both sides with the vectors $\mathbf{v}_i$ in the orthonormal basis we find that $c_i = \langle \mathbf{w}, \mathbf{v}_i \rangle$ for all i. On the other hand, the displayed formula implies that

$$|\mathbf{w}|^2 = \sum_i c_i^2$$

so if we substitute the values for c_i as above we obtain the formula to be shown in the exercise.■

15. (a) By definition the projection is defined by

$$P(\mathbf{w}) = \frac{\mathbf{w} \cdot (1, 2, -1)}{6} \cdot (1, -2, 1)$$

and if we write $\mathbf{w} = (x, y, z)$ the right hand side becomes

$$\frac{x_2 y - z}{6} \cdot (1, 2, -1) \ .$$

(b) We find the matrix by evaluating the preceding expression at the standard three unit vectors and writing the resulting three vectors in column form. Here is the numerical answer:

$$\frac{1}{6} \cdot \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

16. By the reasoning of Exercise 14 we know that

$$|\mathbf{x} + \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 = |\mathbf{x}|^2 + |-\mathbf{y}|^2 = |\mathbf{x} - \mathbf{y}|^2$$

so the lengths are equal. On the other hand, since $\mathbf{x}$ and $\mathbf{y}$ are orthogonal we have

$$\langle \mathbf{x} + \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle = |\mathbf{x}|^2 - |\mathbf{y}|^2$$

and since $\mathbf{x}$ and $\mathbf{y}$ have the same lengths this expression is equal to zero, proving orthogonality.■

I.2 : Cross products

1. The cross product is given by the formal determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ 2 & 1 \end{vmatrix} \cdot \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \cdot \mathbf{j} + \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} \cdot \mathbf{k}$$

which is equal to $-\mathbf{i} - \mathbf{j} - \mathbf{k} = (-1, -1, -1)$.■

2. The cross product is given by the formal determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & -3 & 0 \\ -2 & 5 & 0 \end{vmatrix} = \begin{vmatrix} -3 & 0 \\ -5 & 0 \end{vmatrix} \cdot \mathbf{i} - \begin{vmatrix} 12 & 0 \\ -2 & 0 \end{vmatrix} \cdot \mathbf{j} + \begin{vmatrix} 12 & -3 \\ -2 & 5 \end{vmatrix} \cdot \mathbf{k}$$

which is equal to $54\mathbf{k} = (0, 0, 54)$.$\blacksquare$

3. The cross product is given by the formal determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} \quad = \quad \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \cdot \mathbf{i} \; - \; \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \cdot \mathbf{j} \; + \; \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \cdot \mathbf{k}$$

which is equal to $-2\mathbf{i} + 3\mathbf{j} - \mathbf{k} = (-2, 3, -1)$.$\blacksquare$

4. The box product is the 3×3 determinant

$$\begin{vmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad = \quad 6 \;.\blacksquare$$

5. We already know that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \quad = \quad (\mathbf{a} \cdot \mathbf{c})\mathbf{b} \; - \; (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

and hence we also have

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \quad = \quad -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \quad = \quad \mathbf{c} \times (\mathbf{b} \times \mathbf{a}) \quad = \quad (\mathbf{a} \cdot \mathbf{c})\mathbf{b} \; - \; (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \; .$$

To solve this problem and others which follow, we merely substitute for the three vectors in question.

For this problem we have $\mathbf{a} = (2, 0, 1)$, $\mathbf{b} = (0, 3, 0)$, and $\mathbf{c} = (0, 0, 1)$. It follows that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \quad = \quad 1 \cdot (0, 3, 0) \; - \; 0 \cdot (0, 0, 1) \quad = \quad (0, 3, 0) \; .$$

Likewise, we have

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \quad = \quad 1 \cdot (0, 3, 0) \; - \; 0 \cdot (2, 0, 1) \quad = \quad (0, 3, 0) \;.\blacksquare$$

6. The box product is the 3×3 determinant

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \quad = \quad 2 \;.\blacksquare$$

7. For this problem we have $\mathbf{a} = (1, 1, 0)$, $\mathbf{b} = (0, 1, 1)$, and $\mathbf{c} = (1, 0, 1)$. It follows that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \quad = \quad 1 \cdot (0, 1, 1) \; - \; 1 \cdot (1, 0, 1) \quad = \quad (-1, 1, 0) \; .$$

Likewise, we have

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \quad = \quad 1 \cdot (0, 1, 1) \; - \; 1 \cdot (1, 1, 0) \quad = \quad (-1, 1, 0) \;.\blacksquare$$

8. The box product is the 3×3 determinant

$$\begin{vmatrix} 1 & 3 & 1 \\ 0 & 5 & 5 \\ 4 & 0 & 4 \end{vmatrix} \quad = \quad 20 \; + \; 60 \; - \; 20 \quad = \quad 60 \;.\blacksquare$$

9. For this problem we have $\mathbf{a} = (1,3,1)$, $\mathbf{b} = (0,5,5)$, and $\mathbf{c} = (4,0,4)$. It follows that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \quad = \quad 8 \cdot (0,5,5) \;-\; 20 \cdot (4,0,4) \quad = \quad (-80, 40, -80) \; .$$

Likewise, we have

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \quad = \quad 8 \cdot (0,5,5) \;-\; 20 \cdot (1,3,1) \quad = \quad (-20, -100, 20)$$

and thus we FINALLY have an example where the two products $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ are unequal.■

10. Let C be the 2×3 matrix whose first and second rows are $\mathbf{a}$ and $\mathbf{b}$ respectively. Then the set of all vectors perpendicular to $\mathbf{a}$ and $\mathbf{b}$ corresponds to the solution subspace for the equation $CX = 0$ (we say "corresponds to"because the solutions are really 3×1column vectors and we are using the obvious identification of the latter with $\mathbf{R}^3$. The rank of this matrix is 2, so the dimension of the space of solutions is equal to 1 by fundamental results from linear algebra. Therefore, if $\mathbf{w}$ is a nonzero vector which is perpendicular to $\mathbf{a}$ and $\mathbf{b}$, then every other vector $\mathbf{v}$ which is perpendicular to the latter must be a scalar multiple of $\mathbf{w}$, and if $\mathbf{v}$ is nonzero it must be a nonzero multiple of $\mathbf{w}$. But we know that $\mathbf{z} \times \mathbf{b}$ is a nonzero vector which is perpendicular to $\mathbf{a}$ and $\mathbf{b}$, so in particular, if $\mathbf{v}$ is as before then it must be a nonzero multiple of $\mathbf{a} \times \mathbf{b}$.■

11. Following the hint, we shall begin with the Jacobi identity

$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \;+\; \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \;+\; \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) \quad = \quad 0$$

and then we shall subtract the second and third terms from both sides to obtain the following:

$$D(\mathbf{a} \times \mathbf{b}) \quad = \quad \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \quad = \quad -\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \;-\; \mathbf{b} \times (\mathbf{c} \times \mathbf{a})$$

Since a cross product changes sign if the order of the factors is reversed, the right hand side of the preceding equation is equal to

$$\mathbf{a} \times (\mathbf{c} \times \mathbf{b}) \;+\; (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} \quad = \quad \mathbf{a} \times D(\mathbf{b}) \;+\; D(\mathbf{a}) \times \mathbf{b}$$

which is the identity to be verified in the exercise.■

I.3 : Linear varieties

1. By the definitions, we know that $(2,3,1)$ lies on both lines.■

2. Let us consider the general problem of finding a common point on two lines given by the vector pairs $(\mathbf{a}_1, \mathbf{b}_1)$ and $(\mathbf{a}_2, \mathbf{b}_2)$. The two lines have a point in common if and only if there are scalars t and u such that

$$\mathbf{a}_1 \;+\; t\,\mathbf{b}_1 \quad = \quad \mathbf{a}_2 \;+\; u\,\mathbf{b}_2 \; .$$

One can rephrase this to say that a common point exists if and only if $\mathbf{a}_1 - \mathbf{a}_2$ is a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$. Now the latter two vectors are linearly independent in this problem, and if we let B_i and A_i be the column vectors associated to $\mathbf{a}_i$ and $\mathbf{b}_i$, then the lines have a point in common if and only if the system of three equations in two unknowns

$$(B_1 \; B_2) \cdot X \quad = \quad A_1 - A_2$$

has a solution. Since B_1 and B_2 are linearly independent, the standard results on solutions of linear equations systems implies that the system above has a solution if and only if the rank of the 3×3 matrix

$$\left(\, B_1 \; B_2 \;\; [A_1 - A_2] \, \right)$$

is equal to 2.

If we apply this to the specific example in the problem, we have $\mathbf{b}_1 = (3, -1, 1)$, $\mathbf{b}_2 = (4, 1, -3)$ and $\mathbf{a}_1 - \mathbf{a}_2 = (-1, 4, 2)$, so that we obtain the following matrix:

$$\begin{pmatrix} 3 & 4 & -1 \\ -1 & 1 & 4 \\ 1 & -3 & 2 \end{pmatrix}$$

The determinant of this matrix turns out to be nonzero, and therefore the matrix has rank 3. By the preceding discussion, it follows that the two lines cannot have a point in common.■

3. We start with the same general discussion as in the previous exercise, but here the relevant vectors are given as follows:

$$\begin{aligned} \mathbf{b}_1 &= (2, 5, -1) \\ \mathbf{b}_2 &= (-2, 1, 2) \\ \mathbf{a}_1 &= (3, -2, 1) \\ \mathbf{a}_2 &= (7, 8, -1) \\ \mathbf{a}_1 - \mathbf{a}_2 &= (4, 10, 0) \end{aligned}$$

Therefore the crucial 3×3 matrix is given as follows:

$$\begin{pmatrix} 2 & -2 & 4 \\ 5 & 1 & 10 \\ -1 & 2 & 0 \end{pmatrix}$$

Once again the determinant of the matrix is nonzero, so the lines do not have any points in common.■

4. Since $(0, 0, 0)$ lies on this plane, it is a 2–dimensional vector subspace, and a spanning set is given by the remaining two points $(1, 2, 3)$ and $(-2, 3, 3)$. Since these two vectors are linearly independent, it follows that the plane we are trying to describe is the set of all vectors (x, y, z) which are perpendicular to the cross product $(1, 2, 3) \times (-2, 3, 3)$; therefore it is the set of all (x, y, z) satisfying the equation

$$\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ -2 & 3 & 3 \end{vmatrix} \quad = \quad 0 \ .$$

We can rewrite this in the more traditional form $-3x - 9y + 7 = 0$. As usual, one should check the correctness of the calculations with a verification that the original points all lie on this plane.■

5. More generally, if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are noncollinear points in $\mathbf{R}^3$, then the plane containing them is given by the box product equation

$$\left[\mathbf{b} - \mathbf{a}; \mathbf{c} - \mathbf{a}; \mathbf{x} - \mathbf{a} \right] \quad = \quad 0 \ .$$

In this problem we have $\mathbf{a} = (1,2,3)$, $\mathbf{b} = (3,2,1)$, and $\mathbf{c} = (-1,-2,2)$. Therefore the equation for the plane becomes

$$\begin{vmatrix} (x-1) & (y-2) & (z-3) \\ 2 & 0 & -2 \\ -2 & -4 & -1 \end{vmatrix} \quad = \quad 0 \; .$$

If we expand the determinant and simplify, we obtain the equation $-8x + 6y - 8z = -20$, which we may also rewrite as $8x - 6y + 8z = 20$. Once again, direct calculations verify that the three original points all lie on this plane.∎

6. A plane which is parallel to the xy–plane has an equation of the form $z = K$, where K is some constant. The plane containing a point (p,q,r) will have the form $z = r$. In this case $r = 3$, so the equation for the plane in this problem is $z = 3$.∎

7. The plane has an equation of the form $Ax + By + Cz + D = 0$, where we know that (A,B,C) is perpendicular to both $(-2,1,1)$ and $(-3,4,-1)$. Therefore, up to multiplication by a nonzero scalar we know that (A,B,C) is the cross product of the given two vectors. This cross product is equal to $(-5,-5,-5)$, and thus we know that $A = B = C$. Dividing the equation by A, we see that the plane has an equation of the form $x + y + z + D' = 0$. We can evaluate D' by evaluating $x + y + z$ at any point of the plane; in particular, if we do so at the point $(1,4,0)$ which lies on L (hence also on P), we see that $D' = -5$ and hence the equation of the plane is $x + y + z = 5$. One can check directly that all points of both L and M lie on this plane.∎

8. Probably the simplest way to do this is to use row reduction to find the solutions for the system of equations determined by the intersecting planes. In other words, we need to put the augmented matrix

$$\begin{pmatrix} 5 & -3 & 1 & 4 \\ 1 & 4 & 7 & 1 \end{pmatrix}$$

into row reduced echelon form using elementary row operations; the new matrix will have the same solutions. The row reduced echelon matrix obtained in this fashion is equal to

$$\frac{1}{23} \cdot \begin{pmatrix} 23 & 0 & 25 & 19 \\ 0 & 23 & 34 & 1 \end{pmatrix} \; .$$

This yields the equations

$$x \quad = \quad \frac{19 - 25z}{23} \qquad\qquad y \quad = \quad \frac{1 - 34z}{23}$$

and if we set $t = 23z$, then the line is the set of all points of the form

$$\left(\frac{19}{23}, \frac{1}{23}, 0\right) \; + \; t\,(-25,-34,1)$$

where t runs through all possible scalars.∎

9. On an elementary level, this can be seen by noting that if $\mathbf{a}$ and $\mathbf{p}$ are linearly independent, then the 2×2 matrix whose rows are these vectors will have a nonzero determinant, and hence the system of equations $\mathbf{a} \cdot \mathbf{x} = b$, $\mathbf{p} \cdot \mathbf{x} = q$ has a solution by Cramer's Rule. Thus the only way there can be no solutions is if the vectors $\mathbf{a}$ and $\mathbf{p}$ are linearly dependent.∎

10. A linear variety is given by a system of linear equations. If varieties V_1 and V_2 are given by such lists of equations, then their intersection is defined by the linear equations on both of these lists, and hence it is also a linear variety.∎

11. Suppose that $\mathbf{x}$ lies in the intersection. Then we may write $H = \mathbf{x}+V$ and $K = \mathbf{x}+W$, where V and W are $(n-1)$–dimensional vector subspaces, and it follows that

$$H \cap K \;=\; \big(\mathbf{x}+V\big) \cap \big(\mathbf{x}+W\big) \;=\; \mathbf{x} + (V \cap W)\ .$$

If $n \geq 3$, then the dimension formula for intersections of vector subspaces implies that $\dim(V \cap W) \geq 1$. If U is a 1–dimensional subspace of $V \cap W$, then the line $\mathbf{x}+U$ is contained in

$$\mathbf{x} + (V \cap W) \;=\; H \cap K\ .$$

If we also have $N \geq 4$, then the dimension formula implies $\dim(V \cap W) \geq 2$. If U is a 2–dimensional subspace of $V \cap W$, then the plane $\mathbf{x}+U$ is contained in $\mathbf{x} + (V \cap W) = H \cap K$.∎

12. Suppose that $\mathbf{x} \in S \cup T$. Then either $(\mathbf{a}_i \cdot \mathbf{x}) - b_i \;=\; 0$ for all i or else $(\mathbf{c}_j \cdot \mathbf{x}) - d_j \;=\; 0$ for all j. In either case the products

$$(\mathbf{a}_i \cdot \mathbf{x} - b_i) \cdot (\mathbf{c}_j \cdot \mathbf{x} - d_j) \;=\; 0$$

for all i and j, and hence $S \cup T$ is contained in the set of all points satisfying these equations. Conversely, suppose that $\mathbf{x}$ does not lie in $S \cup T$. Since $\mathbf{x}$ does not lie in S, it follows that there is some integer P for which $(\mathbf{a}_P \cdot \mathbf{x} - b_P) \neq 0$. Likewise, since $\mathbf{x}$ does not lie in T there is some integer Q such that $(\mathbf{c}_Q \cdot \mathbf{x} - d_Q) \neq 0$. If we multiply these equations together we find that

$$(\mathbf{a}_P \cdot \mathbf{x} - b_P) \cdot (\mathbf{c}_Q \cdot \mathbf{x} - d_Q) \;\neq\; 0$$

which means that $\mathbf{x}$ does not lie in the set of points satisfying the given set of equations.∎

13. Suppose first that the points $\{P_1, \cdots, P_n\}$ are coplanar. Suppose that the plane containing them is defined by the equation $Kx + Ly + Mz + N = 0$, where $(K, L, M) \neq (0,0,0)$. Let W be the vector subspace of $\mathbf{R}^4$ consisting of all vectors which are perpendicular to (K, L, M, N), or equivalently to the 1-dimensional subspace spanned by that vector. Then as in the proof of Proposition 2, it follows that W is 3-dimensional. Furthermore, since the vectors P_i lie on the given plane, it follows that the vectors $\mathbf{q}_i$ belong to the subspace W, and hence the subspace W' spanned by the vectors $\mathbf{q}_i$, so that W' is contained in the proper subspace W and hence must be a proper subspace itself.

Conversely, suppose that the span W' is a proper subspace, and let $U = (W')^{\perp}$. As in the proof of Proposition 3, we know that U has a positive dimension and hence contains some nonzero vector (K, L, M, N). It follows that

$$Ka_i \;+\; Lb_i \;+\; Mc_i \;+\; N \;=\; 0, \quad \text{where} \quad 1 \;\leq\; i \;\leq\; n\ .$$

If one of K, L, M is nonzero, then these equations imply that the points P_i lie on the plane defined by the equation $Kx + Ly + Mz + N = 0$.

Suppose now that $K = L = M = 0$; it follows that N must be nonzero. However, now the displayed equations become

$$N \;=\; 0 \cdot a_i \;+\; 0 \cdot b_i \;+\; 0 \cdot c_i \;+\; N \;=\; 0$$

which contradicts the conclusion of the preceding sentence. This means that at least one of K, L, M must be nonzero, and by the preceding paragraph this means that the given points are coplanar.∎

NOTE. The preceding exercise yields a computational test to determine whether a finite set of points is coplanar. If the original points are given by P_i, let $\mathbf{q}_i$ be constructed as in the preceding solution, and let A be the $n \times 4$ matrix whose rows are the vectors $\mathbf{q}_i$. Then the original set of vectors is coplanar if and only if the row-reduced echelon form of A has at most 3 nonzero rows.