SOLUTIONS TO EXERCISES FOR

MATHEMATICS 133 — Part 2a

Winter 2009

NOTE ON ILLUSTRATIONS. Drawings for several of the solutions in this file are available in the following document:

`http://math.ucr.edu/~res/math133/math133solutions2afigures.pdf`

I. Topics from linear algebra

I.4 : Barycentric coordinates

1. The general method is to take a typical point $\mathbf{x}$ and write $\mathbf{x}-\mathbf{a}$ as a linear combination

$$\mathbf{x}-\mathbf{a} \quad = \quad u\mathbf{b}-\mathbf{a} \; - \; v\mathbf{c}-\mathbf{a}$$

from which one obtains the barycentric coordinate expression

$$\mathbf{x} \quad = \quad (1-u-v)\mathbf{a} \; + \; u\mathbf{b} \; + \; v\mathbf{c} \; .$$

For all the problems below we have $\mathbf{b}-\mathbf{a} = (2,0)$ and $\mathbf{c}-\mathbf{a} = (1,1)$.

(a) In this case $\mathbf{x}-\mathbf{a} = (1,0) = \frac{1}{2}\cdot(2,0)+0\cdot(1,1)$. Hence $u = \frac{1}{2}$, $v = 0$, and t must be equal to $\frac{1}{2}$.

(b) In this case $\mathbf{x}-\mathbf{a} = (2,1) = \frac{1}{2}\cdot(2,0)+1\cdot(1,1)$. Hence $u = \frac{1}{2}$, $v = 1$, and t must be equal to $-\frac{1}{2}$.

(c) In this case $\mathbf{x}-\mathbf{a} = (\sqrt{2}+1,\sqrt{2}) = \frac{1}{2}\cdot(2,0)+\sqrt{2}\cdot(1,1)$. Hence $u = \frac{1}{2}$, $v = \sqrt{2}$, and t must be equal to $\frac{1}{2}-\sqrt{2}$.

(d) In this case $\mathbf{x}-\mathbf{a} = (1,5) = -2\cdot(2,0)+5\cdot(1,1)$. Hence $u = -2$, $v = 5$, and t must be equal to -2.

(e) In this case $\mathbf{x}-\mathbf{a} = (3,-1) = 2\cdot(2,0)-1\cdot(1,1)$. Hence $u = 2$, $v = -1$, and t must be equal to 0.

(f) In this case $\mathbf{x}-\mathbf{a} = (\frac{1}{2},-\frac{1}{2}) = \frac{1}{2}\cdot(2,0)-\frac{1}{2}\cdot(1,1)$. Hence $u = \frac{1}{2}$, $v = -\frac{1}{2}$, and t must be equal to 1.■

2. We shall first show that if $\{\,\mathbf{v}_0,\ \cdots\ ,\mathbf{v}_n\,\}$ is affinely independent then the associated set $\{\,\mathbf{v}_1-\mathbf{v}_0,\ \cdots\ ,\mathbf{v}_n-\mathbf{v}_0\,\}$ is linearly independent. — Suppose we have

$$\sum_{j=1}^{n} c_j(\mathbf{v}_j-\mathbf{v}_0) \quad = \quad \mathbf{0} \; .$$

If we add $\mathbf{v}_0$ to both sides and rearrange terms, we obtain

$$\mathbf{v}_0 \quad = \quad \mathbf{v}_0 \; + \; \sum_{j=1}^{n} \; c_j(\mathbf{v}_j - \mathbf{v}_0) \quad = \quad \left(1 \; - \; \sum_{j=1}^{n} \; c_j\right) \mathbf{v}_0 \; + \; \sum_{j=1}^{n} \; c_j \mathbf{v}_j \; .$$

Now the left hand side is an expression of $\mathbf{v}_0$ as a linear combination of $\{\, \mathbf{v}_0, \; \cdots \; , \mathbf{v}_n \,\}$ such that the coeeficients add up to 1, and therefore by the affine independence assumption we know that the corresponding coefficients on the left and right hand sides of the displayed equation(s) are equal. In particular, this means that $c_j = 0$ for all j; the latter in turn implies that the set $\{\, \mathbf{v}_1 - \mathbf{v}_0, \; \cdots \; , \mathbf{v}_n - \mathbf{v}_0 \,\}$ is linearly independent.

Conversely, suppose $\{\, \mathbf{v}_1 - \mathbf{v}_0, \; \cdots \; , \mathbf{v}_n - \mathbf{v}_0 \,\}$ is linearly independent, and suppose that $\mathbf{x}$ is an affine combination $\sum_j \; t_j \mathbf{v}_j$, where $\sum_j t_j = 1$. We then have

$$\mathbf{x} \; - \; \mathbf{v}_0 \quad = \quad \left(\sum_{j=0}^{n} \; t_j \mathbf{v}_j\right) \; - \; \mathbf{v}_0 \quad = \quad \left(\sum_{j=0}^{n} \; t_j \mathbf{v}_j\right) \; - \; \left(\sum_{j=0}^{n} t_j \; \mathbf{v}_0\right) \quad = \quad \sum_{j=1}^{n} \; t_j(\mathbf{v}_j - \mathbf{v}_0).$$

If we now take an arbitrary expression of $\mathbf{x}$ as an affine combination $\sum_j \; u_j \mathbf{v}_j$, where $\sum_j u_j = 1$, then the same sort of argument implies that

$$\mathbf{x} \; - \; \mathbf{v}_0 \quad = \quad \sum_{j=1}^{n} \; u_j(\mathbf{v}_j - \mathbf{v}_0)$$

and by the linear independence of $\{\, \mathbf{v}_1 - \mathbf{v}_0, \; \cdots \; , \mathbf{v}_n - \mathbf{v}_0 \,\}$ we therefore know that $u_j = t_j$ for all $j \geq 1$. But then we also have

$$t_0 \quad = \quad 1 \; - \; \sum_{j=1}^{n} \; t_j \quad = \quad 1 \; - \; \sum_{j=1}^{n} \; u_j \quad = \quad u_0$$

so that all the corresponding coefficients t_j and u_j are equal and hence the set $\{\, \mathbf{v}_0, \; \cdots \; , \mathbf{v}_n \,\}$ is affinely independent.■

3. We shall follow the hint. The lines **ab** and **cd** are given by $\mathbf{a} + V$ and $\mathbf{c} + V$ respectively since $\mathbf{b} - \mathbf{a}$ and $\mathbf{d} - \mathbf{c}$ are nonzero scalar multiples of each other, and similarly the lines **ad** and **bc** are given by $\mathbf{a} + W$ and $\mathbf{b} + W$ respectively. By the Coset Property, if either of these pairs of lines has a point in common, then the two lines in the pair must be the same. However, this is impossible in both cases. Since **a**, **d** and **b** are noncollinear, it follows that the lines **ab** and **cd** cannot be equal, and likewise the lines **ad** and **bc** cannot be equal. Hence it follows that **ab** is parallel to **cd** and **ad** is parallel to **bc**.■

4. By the results of the previous problem we know that $C = B + D - A$. Since E is a midpoint we know that $E = \frac{1}{2}(A + B)$. Since F lies on DE and AC we may write

$$tA \; + \; (1-t)C \quad = \quad F \quad = \quad uE \; + \; (1-u)D$$

and if we substitute for C and E in the left and right sides we obtain the following equation:

$$(2t-1)A + (1-t)B \; + \; (1-t)D \quad = \quad tA \; + \; (1-t)C \quad = \quad \frac{u}{2}A \; + \; \frac{u}{2}B \; + \; (1-u)D$$

Since A, B and D are noncollinear and the coefficients on both sides of the equation add up to 1, we may set the corresponding coefficients equal and conclude that $2t - 1 = \frac{1}{2}u$, $1 - t = \frac{1}{2}u$, and $1 - t = 1 - u$. Solving these equations, we see that $t = u = \frac{2}{3}$, so that $F - C = \frac{2}{3}(A - C)$ and $F - D = \frac{2}{3}(E - D)$. Further algebraic manipulation shows that $A - F = (A - C) - (F - C)$ is equal to $\frac{1}{3}(A - C)$ and likewise $E - F = (E - D) - (F - D)$ is equal to $\frac{1}{3}(E - D)$. The distance relationships follow by taking the lengths of the vectors on both sides of the resulting two equations.■

5. By assumption we know that the barycentric coordinate expression for each point $\mathbf{p}_j$ is given by the formula

$$\mathbf{p}_j \quad = \quad t_j\mathbf{a} \; + \; \mathbf{u}_j\mathbf{b} \; + \; \mathbf{v}_j\mathbf{c} \; .$$

Suppose that the three points are collinear, so that $\mathbf{p}_3$ lies on the line $\mathbf{p}_1\mathbf{p}_2$. Then we have

$$\mathbf{p}_3 \quad = \quad w\mathbf{p}_2 \; + \; (1 - w)\mathbf{p}_1$$

for some scalar w. Combining this with the previous expansions for $\mathbf{p}_1$ and $\mathbf{p}_2$, we obtain the following expression for $\mathbf{p}_3$:

$$t_3\mathbf{a} \; + \; \mathbf{u}_3\mathbf{b} \; + \; \mathbf{v}_3\mathbf{c} \quad = \quad \mathbf{p}_3 \quad = \quad w\big(t_1\mathbf{a} \; + \; \mathbf{u}_1\mathbf{b} \; + \; \mathbf{v}_1\mathbf{c}\big) \; + \; (1 - w)\big(t_2\mathbf{a} \; + \; \mathbf{u}_2\mathbf{b} \; + \; \mathbf{v}_2\mathbf{c}\big) \quad =$$

$$[wt_2 + (1 - w)t_1]\mathbf{a} \; + \; [wu_2 + (1 - w)u_1]\mathbf{b} \; + \; [wv_2 + (1 - w)v_1]\mathbf{c}$$

The coefficients of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in the last expression add up to 1, and thus we may equate the barycentric coordinates in the first and last expressions in the chain of equations displayed above. We may rewrite these equations for barycentric coordinates in vector form as follows:

$$\big(t_3, u_3, v_3\big) \quad = \quad w \cdot \big(t_2, u_2, v_2\big) \; + \; (1 - w) \cdot \big(t_3, u_3, v_3\big)$$

Therefore the row of the matrix

$$A \quad = \quad \begin{pmatrix} t_1 & u_1 & v_1 \\ t_2 & u_2 & v_2 \\ t_3 & u_3 & v_3 \end{pmatrix}$$

are linearly dependent, and hence the determinant of this matrix is equal to zero.

Conversely, suppose that the determinant of the matrix A is zero. Then the rows are linearly independent, so there are scalars x, y, z not all zero such that

$$x \cdot \big(t_3, u_3, v_3\big) \; + \; y \cdot \big(t_2, u_2, v_2\big) \; + \; z \cdot \big(t_3, u_3, v_3\big) \quad = \quad \mathbf{0} \; .$$

Since the three coordinates of the expression on the left hand side are all equal to zero, we have the following equations:

$$xt_1 \; + \; yt_2 \; + \; zt_3 \quad = \quad 0$$

$$xu_1 \; + \; yu_2 \; + \; zu_3 \quad = \quad 0$$

$$xv_1 \; + \; yv_2 \; + \; zv_3 \quad = \quad 0$$

These equations in turn imply the vector equation

$$x\mathbf{p}_1 \; + \; y\mathbf{p}_2 \; + \; z\mathbf{p}_3 \quad = \quad \mathbf{0} \; .$$

If we add the three scalar equations we obtain

$$0 \quad = \quad (xt_1 \; + \; yt_2 \; + \; zt_3) \; + \; (xu_1 \; + \; yu_2 \; + \; zu_3) \; + \; (xv_1 \; + \; yv_2 \; + \; zv_3) \quad =$$

$$x(t_1 \ + \ u_1 \ + \ v_1) \ + \ y(t_2 \ + \ u_2 \ + \ v_2) \ + \ z(t_3 \ + \ u_3 \ + \ v_3)$$

and since $t_1 \ + \ u_1 \ + \ v_1 \ = \ t_2 \ + \ u_2 \ + \ v_2 \ = \ t_3 \ + \ u_3 \ + \ v_3 \ = \ 1$ the preceding equations imply that $x + y + z = 0$.

We know that at least one of $x, \ y, \ z$ is nonzero. Suppose that $x \neq 0$; then it follows that $-x^{-1}y - x^{-1}z = 1$ and $\mathbf{p}_1 = -x^{-1}y\mathbf{p}_2 - x^{-1}z\mathbf{p}_3$, so that $\mathbf{p}_1$ lies on the line containing $\mathbf{p}_2$ and $\mathbf{p}_3$. Similarly, if $y \neq 0$ then it follows that $\mathbf{p}_2$ lies on the line containing $\mathbf{p}_1$ and $\mathbf{p}_3$, and finally if $z \neq 0$ then it follows that $\mathbf{p}_3$ lies on the line containing $\mathbf{p}_1$ and $\mathbf{p}_2$. In all three cases the points $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are collinear.■

6. Following the hint, we shall use the criterion of the previous exercise. With the hypotheses in Menelaus' Theorem, we obtain the following criterion for D, E and F to be collinear:

$$0 \ = \ \begin{vmatrix} t & (1-t) & 0 \\ 0 & u & (1-u) \\ (1-v) & 0 & v \end{vmatrix} \ = \ tuv \ + \ (1-t)(1-u)(1-v)$$

Since the vanishing of the right hand side is equivalent to the condition in the Theorem of Menelaus, this proves the latter.■

7. In this problem the hypothesis on D is equivalent to the equation $t = -1$, while the hypothesis on E is equivalent to the equation $u = \frac{1}{2}$. Therefore the equation in the Theorem of Menelaus becomes

$$\begin{aligned} u \ &= \ -(-1)(1-u)\tfrac{1}{2}v \\ (-1) \cdot \tfrac{1}{2}v \ &= \ -2 \cdot \tfrac{1}{2}(1-v) \ = \ v - 1 \end{aligned}$$

which simplifies to $v \ = \ \frac{2}{3}$. Substituting this into the equation for F in Exercise 6, we get $F = \frac{1}{3}A + \frac{2}{3}C$.■

8. The hypotheses guarantee that none of the numbers $t, \ u, \ v$ is equal to 0 and 1. Furthermore, if we write the intersection point G in the forms

$$xB \ + \ (1-x)E \ = \ yC \ + \ (1-y)F$$

then the condition $G \neq B, C$ implies that neither x nor y is equal to 1.

The hypotheses and elementary algebra also yield expressions for G as an affine combination of $A, \ B, \ C$ by the following chain of equations:

$$G \ = \ xB \ + \ (1-x)E \ = \ xB \ + \ (1-x)uC \ + \ (1-x)(1-u)A$$

$$G \ = \ yC \ + \ (1-y)F \ = \ yC \ + \ v(1-y)A \ + \ (1-x)(1-y)B$$

It follows that the corresponding barycentric coordinates in the right hand expressions equal, so that we have the following relationships among the various coefficients:

$$y \ = \ (1-x)u \qquad x \ = \ (1-v)(1-y) \qquad v(1-y) \ = \ (1-x)(1-u)$$

As noted in the hint, the lines AD, BE and CF are concurrent if and only if G lies on AD, or equivalently the points A, G and D are collinear. By the formula of Exercise 5, this happens if and only if we have

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & t & (1-t) \\ (1-x)(1-u) & x & (1-x)u \end{vmatrix} \ = \ 0 \ .$$

Evaluating the determinant, we see that concurrence is equivalent to the equation $tu(1-x) = (1-v)(1-y)(1-t)$. We have already seen that $v(1-y) = (1-x)(1-u)$, and we know that all the factors are nonzero (hence both sides are nonzero); therefore the equation in the previous sentence is equivalent to

$$tuv(1-x)(1-y) \quad = \quad (1-t)(1-u)(1-v)(1-x)(1-y)$$

and since $(1-x)$ and $(1-y)$ are nonzero it follows that the equation displayed above is equivalent to $tuv = (1-t)(1-u)(1-v)$, which is the criterion stated in the theorem.■

9. In this problem we have $D = \frac{1}{2}(B+C)$ and $E = uC + (1-u)A$, $F = uB + (1-u)A$. Since G lies on the lines AD, BE, and CF, we have an equation of the form

$$G \quad = \quad sA \; + \; \frac{1-s}{2}B \; + \; \frac{1-s}{2}C \quad = \quad rB \; + \; u(1-r)C \; + \; (1-u)(1-r)A$$

for suitably chosed real numbers r and s. Both the second and the third expression are affine combinations of G in terms of A, B and C, and therefore the corresponding coefficients must be equal. Thus we have

$$\frac{1-s}{2} \quad = \quad r \;\; \text{and} \;\; u(1-r) \quad = \quad \frac{1-s}{2} \quad = \quad r \; .$$

The second string of equations yields

$$r \quad = \quad \frac{u}{1+u}$$

while the first yields

$$s \quad = \quad 1 - 2r \quad = \quad \frac{1-u}{1+u} \; .$$

If we substitute these quantities into the expression for G and also use the equation for D in the first sentence of this solution, we conclude that

$$G \quad = \quad \frac{1-u}{1+u}A \; + \; \frac{2u}{1+u}D \; .■$$

10. By hypothesis, for each $i = 0, \; \cdots, m$ we have $\mathbf{w}_i = \sum_j \; c_{i,j}\mathbf{v_j}$ where the sum runs from $j = 0$ to $j = k$ and $\sum_j \; c_{i,j} = 1$ for each i. Suppose now that we may write $\mathbf{y}$ as an affine combination $\sum_i \; t_i\mathbf{w}_i$, where the sum runs from $i = 0$ to m and $\sum \; t_i = 1$. Then we have

$$\mathbf{y} \quad = \quad \sum_{i=0}^{m} \left(\sum_{j=0}^{k} \; c_{i,j}\mathbf{v}_j \right) \quad = \quad \sum_{i,j=(0,0)}^{(m,k)} \; t_i c_{i,j}\mathbf{v}_j \quad = \quad \sum_{j=0}^{k} \left(\sum_{i-0}^{m} \; t_i c_{i,j} \right) \mathbf{v}_j$$

and to show this linear combination of the $\mathbf{v}_j$'s is an affine combination we need to show that

$$\sum_{j=0}^{k} \left(\sum_{i=0}^{m} \; t_i c_{i,j} \right) \quad = \quad 1 \; .$$

But the left hand side is equal to

$$\sum_{i,j=(0,0)}^{(m,k)} \; t_i c_{i,j} \quad = \quad \sum_{i=0}^{m} \left(\sum_{j=0}^{k} \; c_{i,j} \right)$$

and since each sum $\sum_j c_{i,j} = 1$ the right hand side reduces to $\sum_i t_i$, which we know is equal to 1.■

11. Call the first point D and denote the remaining points by A, B and C respectively. We need to find scalars v and w such that $D - A = v(B - A) + w(C - A)$; in this problem we have $D - A = (1, 0)$, $B - A = (2, 1)$ and $C - A = (2, -1)$. Substituting in the numerical values, we obtain the following equation(s):

$$(1, 0) \quad = \quad v\,(2, 1) \; + \; w\,(2, -1)$$

If we solve these for v and w we obtain $v = \frac{1}{4}$ and $w = \frac{1}{4}$. It follows that if

$$D \quad = \quad u\,A \; + \; v\,B \; + \; w\,C$$

with $u + v + w = 1$, then we must have $D \; = \; \frac{1}{2}\,A \; + \; \frac{1}{4}\,B \; + \; \frac{1}{4}\,C$.■

Yet another problem. Suppose that $A = (0, 0)$, $B = (1, 0)$ and $C = (1, 1)$. Find the barycentric coordinates of $D = (\frac{1}{2}, 2)$ with respect to A, B, C. — SOLUTION: The first step is to express $D - A$ as a linear combination of $B - A$ and $C - A$; in other words, we want y and z such that $(\frac{1}{2}, 2) = y(1, 0) + z(1, 1)$. This is equivalent to the system of equations $\frac{1}{2} = y + z$, $2 = z$; the unique solution for this system is $y = -\frac{3}{2}$ and $z = 2$ (check this out!). To obtain the remaining barycentric coordinate x such that $D = xA + yB = zC$ where $x + y + z = 1$, we substitute into the formula $x = 1 - y - z$ to find that $x = \frac{1}{2}$.■

12. The midpoint is given by

$$\frac{1}{2}\left(B \; + \; (2A - B)\right) \quad = \quad \frac{1}{2}\,(2A) \quad = \quad A\;.$$