SOLUTIONS TO EXERCISES FOR

MATHEMATICS 133 — Part 2b

Fall 2009

NOTE ON ILLUSTRATIONS. Drawings for several of the solutions in this file are available in the following document:

`http://math.ucr.edu/~res/math133/math133solutions2bfigures.pdf`

II. Linear algebra and Euclidean geometry

II.1 : Approaches to Euclidean geometry

1. Two planes are needed. We have noncoplanar lines AB, AC and AD, and they lie in the union of planes ABC and ACD. They cannot lie in one plane because we are assuming the lines are not coplanar.■

2. Again two planes are needed. If the points are A, B, C, D, E then the planes ABC and CDE contain all three points.

It is natural to ask what happens if we have more than five points. If there are six points, then two planes are still enough (the plane of the first three points and the plane of the last three points), while if there are seven points then three planes are needed (if two planes contain a set of seven points, at least one will contain four of them). More generally, the minimum number of planes needed to contain k points, no three of which are collinear, is equal to $\mathbf{INT}\big((n+2)/3\big)$.■

3. The lines **ab** and **cd** are not coplanar since **a, b, c, d** are not coplanar. If the lines intersected, then they would be coplanar, and of course they are not parallel since they are not coplanar.■

4. Suppose that two of the lines are the same, say $\mathbf{xp}_i = \mathbf{xp}_j$. If we call this line M, then M contains $\mathbf{x}$, $\mathbf{p}_i$ and $\mathbf{p}_j$. Now L contains the last two of these three points, therefore $L = M$. Hence we also have $\mathbf{x} \in M = L$; but this contradicts the basic condition $\mathbf{x} \notin L$. The problem with the logic arises from our assumption that $\mathbf{xp}_i = \mathbf{xp}_j$, and thus these lines must be sdistinct.

To prove that there are infinitely many lines through $\mathbf{x}$, by the preceding argument it is enough to show that L contains infinitely many points; specifically, such an infinite family can be found by taking all points of the form $\mathbf{p_1} + n(\mathbf{p}_2 - \mathbf{p}_1)$.■

5. Follow the hint. Let M be a line distinct from $L_1, \cdots, L_n$; this line exists by the previous result. We then have

$$M \cap \big(L_1 \cup \cdots \cup L_n\big) \quad = \quad (M \cap L_1) \cup \cdots \cup (M \cap L_n) \ .$$

Now each set $M \cap L_j$ has either one point or no points, and thus the displayed set contains at most n points. Now M contains infinitely many points, so that there is some point $y \in M$ not in the displayed set. We claim that $y \notin L_j$ for all j. This follows because $y \in M$ and $y \notin M \cap L_j$ for all j. The only way this can happen is if y does not lie on any of the lines L_j.■

6. Again follow the hint, taking the points $\mathbf{p}_j$, a point $\mathbf{x}$ distinct from all of them, and a line L through $\mathbf{x}$ which is not equal to any of the lines $\mathbf{xp}_j$. We claim that L does not contain any of the points $\mathbf{p}_j$. Suppose that some $\mathbf{p}_k \in L$. Since L contains both this point and $\mathbf{x}$, it follows that $L = \mathbf{xp}_k$. But this contradicts our choice of L, so we have a contradiction. The problem arises from our assumption that L was one of the lines $\mathbf{xp}_j$, and hence L must be distinct from all of these lines.■

7. We first show that the planes **abu** and **abv** are distinct. If they are equal, let Q be the common plane they represent. Since **u** and **v** lie on this plan, it follows that the line **uv** is contained in Q. Now the line **uv** contains the point **c**, and therefore Q contains the points **a, b, c**. The latter means that Q is equal to the plane $P = \mathbf{abc}$. Since we also have $\mathbf{z} \in \mathbf{uv}$, it follows that we also have $\mathbf{z} \in Q = P$. This contradicts our assumption that **z** was not on P, and thus the assumption that $\mathbf{abu} = \mathbf{abv}$ must be false. Therefore these two planes must be distinct.

To see that there are infinitely many planes through **ab**, note that there are infinitely many points $\mathbf{u}_j$ on the line **cz** and by the preceding paragraph the planes $\mathbf{abu}_j$ must be distinct.■

8. Let L be a line in P, and let $B,\ C,\ D$ be distinct points on L. Then there is a point A which does not lie on L. By Exercise 4, the lines $AB,\ AC,\ AD$ are distinct. Each of these lines contains a third point, so let $E \in AB$, $F \in AC$ and $G \in AD$ be the third points. These three points are distinct; for example, if $E = F$ then the lines AB and AC would be equal, and since this common line contains B and C it must be equal to L; this cannot happen since $A \in AB$ but $A \notin L$. Likewise (interchanging the roles of $B,\ C,\ D$) we can show that $E \neq G$ and $F \neq G$. By construction we know that A is not equal to E, F or G. We claim that B is not equal to any of these three points either; by construction we cannot have $B = E$. If, say, $B = F$, then we know that the lines AB and $AF = AC$ are equal, so that $C \in AB$and $A \in BC = L$. This contradicts our choice of A, so we cannot have $B = F$. Similarly, we cannot have $B = G$ (switch the roles of C and D in the argument we just finished). Finally, if we switch the roles of B and either C or D, then we may similarly conclude that C and D are also distinct from $E,\ F,\ G$. Putting this all together, we conclude that the points $A,\ B,\ C,\ D,\ E,\ F,\ G$ are all distinct.■

II.2 : Synthetic axioms of order and separation

1. Let $\mathbf{a} = (a_1,\, a_2,\, a_3)$, $\mathbf{b} = (b_1,\, b_2,\, b_3)$, and $\mathbf{c} = (c_1,\, c_2,\, c_3)$. Since the three points are collinear we have $\mathbf{b}_2 = \mathbf{a} + t(\mathbf{c} - \mathbf{a})$ for some scalar t. We need to show that under the conditions of the exercise we have $0 < t < 1$.

If we specialize the formula for $\mathbf{b}_2$ to the first coordinate we find that $b_1 = a_1 + t(c_1 - a_1)$. Solving for t, we obtain the following equation:

$$t \quad = \quad \frac{b_1 - a_1}{c_1 - a_1}$$

Since $0 < b_1 - a_1 < c_1 - a_1$ it follows that $0 < t < 1$ as required.■

2. Following our usual procedure, we need to find scalars s and t such that $0 < t < 1 < s$ and

$$tA \ + \ (1-t)B \quad = \quad sC \ + \ (1-s)D \ .$$

Since $C = \mathbf{0}$, it is easier to work this problem using ordinary coordinates instead of barycentric coordinates. Thus we are looking for a point whose coordinates are $(t, 1-t) = \big(2(s-1),\ (s-1)\big)$. Setting the first and second coordinates equal, we have $t = 2s-2$ and $1-t = s-1$. If we solve this system of equations we obtain the values $t = \frac{2}{3}$ and $s = \frac{4}{3}$. Thus there is a point X such that X is between A and B, and also C is between D and X.■

3. The betweenness conditions in the problem yield the equations $C = A + u(B-A)$ and $E = A + v(B-A)$ where $u,\ v > 1$. Therefore we want to find a point X such that

$$X \ = \ sB \ + \ (1-s)E \ = \ tD \ + \ (1-t)C$$

where $0 < s,\ t < 1$. We shall use the same method as in many other problems. Expand the second and third expressions in terms of $A,\ B$, and D, note that the coefficients add up to one in both cases, and use the uniqueness of barycentric coordinates to conclude that the coefficients of $A,\ B$, and D are equal.

Here is what we get if we expand the two expressions:

$$X \ = \ sB \ + \ (1-s)[A \ + \ v(D-A)] \ = \ (1-s)(1-v)A \ + \ sB \ + \ (1-s)vD$$

$$X \ = \ tD \ + \ (1-t)[A \ + \ u(B-A)] \ = \ (1-t)(1-u)A \ + \ (1-t)uB \ + \ tD$$

The coefficients of $A,\ B$, and D in both right hand expressions add up to 1, and therefore by the uniqueness of barycentric coordinates we know that the corresponding coefficients are equal. Therefore we have

$$s \ = \ (1-t)u, \qquad (1-s)(1-v) \ = \ (1-t)(1-u), \qquad t \ = \ (1-s)v$$

and these yield the following system of two linear equations for s and t:

$$s \ + \ tu \ = \ u$$

$$sv \ + \ t \ = \ v$$

If we solve this system we obtain the following values:

$$s \ = \ \frac{u-uv}{1-uv}, \qquad t \ = \ \frac{v-uv}{1-uv}$$

To complete the proof, we need to check that s and t are between 0 and 1; the key point is to recall that u and v are both greater than 1. The latter inequalities imply that the numerators and denominators of the expressions for s and t are both negative, and thus it follows that the quotient expressions for s and t are both positive. To see that these numbers are less than 1, multiply the numerators and denominators by -1 to obtain

$$s \ = \ \frac{uv-u}{uv-1}, \qquad t \ = \ \frac{uv-v}{uv-1}$$

and note that $0 < uv-u,\ uv-v < uv-1$ because u and v are both greater than 1.■

4. The equation of L is $4y = x+10$, and this may be rewritten in the form $0 = 4y-x-10 = f(x,y)$. Two points (a,b) and (c,d) lie on the same side of L if and only if the signs of $f(a,b)$ and $f(c,d)$ are both positive or both negative.

If we substitute the values for the coordinates of the two points, we obtain $f(4,-2) = 4(-2) - 4 - 10 = -22$ and $f(6,8) = 4\cdot 8 - 6 - 10 = 16$. Since one value is positive and the other is negative, it follows that the two points lie on opposite sides of the line L.■

5. In this case the line L is defined by the equation $3x - y - 7 = 0$. We now have $f(8,5) = 12$ and $f(-2,4) = -17$, so once again the two points lie on opposite sides of the line L.■

6. Suppose that X and Y lie on the line M. Since $L \cap M = \emptyset$, it follows that neither X nor Y lies on L. Suppose that X and Y lie on opposite sides of L. Then by the Plane Separation Property it follows that the segment (XY) meets L in some point W. Since (XY) is a subset of the line M, it follows that $W \in L \cap M$; but this contradicts our original assumption that $L \cap M = \emptyset$. Therefore the only possibility is that X and Y lie on the same side of L.■

7. The analogous result in 3–dimensional space is as follows:

Let P and Q be parallel (hence disjoint) planes in space. Then all points of Q lie on the same side of P.

Here is the analogous proof: Suppose that $X$and Y lie on the plane Q. Since $P \cap Q = \emptyset$, it follows that neither X nor Y lies on P. Suppose that X and Y lie on opposite sides of P. Then by the Plane Separation Property it follows that the segment (XY) meets P in some point W. Since X and Y lie in Q, we know that Q contains the entire line XY, and hence it also contains the segment (XY). Therefore it follows that $W \in P \cap Q$; but this contradicts our original assumption that $P \cap Q = \emptyset$. Therefore the only possibility is that X and Y lie on the same side of P.■

8. Suppose the conclusion is false, so that there is some triangle ΔABC and line L such that L contains points on each of the open segments (AB), (BC) and (AC). We shall call these points X, Y and Z respectively.

Now the line L cannot be equal to any of the lines AB, BC or AC; to see that $L \neq AB$, note that L also contains a point of (AC) and hence we would also have $L = AZ = AC$, which in turn would imply that A, B and C would be collinear. Similar considerations show that L is not equal to either of the lines BC or AC.

We also claim that L cannot contain any of the points A, B, C. For example, if $A \in L$, then $X, Z \in L$ would imply $A = AX = AY$, and since $AX = AB$ and $AZ = AC$ would imply that L contains all of A, B, C and this is impossible since these three points are not collinear. Interchanging the roles of the three vertices, we also find that neither B nor C can lie on L.

Since $X \in L$ and $A * X * B$ our results on order and separation imply that A and B lie on opposite sides of L (compare the proof of Pasch's Theorem). Similarly $Y \in L$ and $B * Y * C$ imply that B and C lie on opposite sides of L, and $Z \in L$ and $A * Z * C$ imply that A and C lie on opposite sides of L. On the other hand, the first two conclusions imply that both A and C both lie on the side of L which does not contain B, so that they must lie on the same side of L, and thus we have a contradiction. The source of this contradiction was our original assumption that there was a triangle ΔABC and line L such that L contains points on each of the open segments (AB), (BC) and (AC). Therefore this cannot happen, and the statement in the exercise must be true for every line and triangle in the same plane.■

9. By assumption we have

$$\mathbf{y} \;=\; \sum_j b_j \mathbf{w}_j \qquad \text{where all} \quad b_j \;\geq\; 0 \quad \text{and} \quad \sum_j b_j \;=\; 1\;.$$

Furthermore, for each j we also know that

$$\mathbf{w}_j \quad = \quad \sum_k \, a_{j,k}\mathbf{v}_k \qquad \text{where all} \quad a_{j,k} \;\geq\; 0 \quad \text{and} \quad \sum_k \, a_{j,k} \;=\; 1 \quad \text{for all } j.$$

Therefore we have

$$\mathbf{y} \quad = \quad \sum_{j,k} \, b_j \, a_{j,k} \, \mathbf{v_k}$$

and if we set $c_k \;=\; \sum_j \; b_j \, a_{j,k}$ it follows immediately that $c_k \geq 0$ for all j and

$$\sum_k \; c_k \quad = \quad \sum_{j,k} \; b_j \, a_{j,k} \quad = \quad \sum_j \; b_j \left(\sum_k \; a_{j,k} \right) \; .$$

Since the summations inside the parenthese are all equal to 1, it follows that the summation of interest is equal to $\sum_j \; b_j$; but we know the latter is equal to 1. Therefore it follows that $\mathbf{y}$ is a convex combination of the vectors $\mathbf{v}_k$ in the set S.■

10. If a line is given by an equation of the form $g(x, y) = Ax + By = C$, then the two sides of this line are defined by the inequalities $g(x, y) < C$ and $g(x, y) > C$. In our case $g(x, y) = 2x + 3y$, so in order to determine which side of the line contains the point $(4, 5)$ we must find out whether $g(4, 5) < 6$ or $g(4, 5) > 6$. Since $2 \cdot 4 + 3 \cdot 5 = 23 > 6$, we know that the second possibility holds.

Next, we must find the values of t for which $g(17, t) > 6$. This inequality amounts to $34+3t > 6$, or equivalently $3t > 28$, or $t > -28/3$.■

11. Let $g(x, y) = 3y - 2x$; we need to determine which of the numbers $g(2, 5), g(6, 5)$ and $g(6, 2)$ are positive and which are negative (the possibility that all are positive or negative cannot yet be dismissed). Computing the three values of the function, we find that $g(2, 5) = 12$, $g(6, 5) = -3$ and $g(6, 2) = -12$, which means that $(6, 5)$ and $(6, 2)$ lie on the same side of the given line, and $(2, 5)$ lies on the opposite side of these points.■

12. Let $C = 2A - B$. By the results of this section, the midpoint is given by

$$\frac{1}{2}(B + C)$$

, and if we compute this out we get

$$\frac{1}{2}\big(\, B \; + \; (2A - B) \, \big) \quad = \quad \frac{1}{2} \, (2A) \quad = \quad A$$

which is what we wanted to verify.■