SOLUTIONS TO EXERCISES FOR

MATHEMATICS 133 — Part 4

Fall 2007

NOTE ON ILLUSTRATIONS. Drawings for several of the solutions in this file are available in the following file:

`http://math.ucr.edu/~res/math133/math133solutions4figures.pdf`

III. Basic Euclidean concepts and theorems

III.1 : Perpendicular lines and planes

1. Assuming that the three planes P, Q and T have one point in common but do not have a line in common, define lines L, M, N such that $L = P \cap Q$, $M = P \cap T$ and $N = Q \cap T$. Then we have $L \cap M \cap N = P \cap Q \cap T$, and we know it contains at least one point. However, if the planes do not have a line in common, then it follows that $L \neq M$, for otherwise they would. Now the lines L and M have at most one point in common, so the same is true for the subset $L \cap M \cap N$. Since this subset contains at least one point and cannot contain more than one point, it must consist of exactly one point.∎

2. As usual, follow the hint. The formula from Section I.2 states that

$$\big(\mathbf{v} \times \mathbf{w}\big) \cdot \big(\mathbf{y} \times \mathbf{z}\big) \;\;=\;\; (\mathbf{v} \cdot \mathbf{y})(\mathbf{w} \cdot \mathbf{z}) \;-\; (\mathbf{v} \cdot \mathbf{z})(\mathbf{w} \cdot \mathbf{y}) \;.$$

In our situation $\mathbf{w} = \mathbf{z} = \mathbf{u}$, while $\mathbf{v} = \mathbf{a}$ and $\mathbf{y} = \mathbf{b}$; we know that $\mathbf{u}$ is a unit vector which is perpendicular to $\mathbf{a}$ and $\mathbf{b}$. If we make these substitutions we find that

$$\big(\mathbf{a} \times \mathbf{u}\big) \cdot \big(\mathbf{b} \times \mathbf{u}\big) \;\;=\;\; \mathbf{a} \cdot \mathbf{b}$$

and also that $|\mathbf{a}| = |\mathbf{a} \times \mathbf{u}|$ as well as $|\mathbf{b}| = |\mathbf{b} \times \mathbf{u}|$. Combining these, we see that the cosine of the angle $\angle(\mathbf{x} + \mathbf{a})\mathbf{x}(\mathbf{x} + \mathbf{b})$ is equal to the cosine of the angle $\angle(\mathbf{a} \times \mathbf{u})\mathbf{0}(\mathbf{b} \times \mathbf{u})$.∎

3. Follow the hint, and write $P = \mathbf{x} + W$, where W is a 2–dimensional vector subspace. Let $\mathbf{e}$ and $\mathbf{f}$ form an orthonormal basis for W, let U and V be the 1–dimensional subspaces they span, and take L and M to be the lines $\mathbf{x} + U$ and $\mathbf{x} + V$. Since $\mathbf{e}$ and $\mathbf{f}$ are perpendicular, the lines L and M will also be perpendicular. Suppose that we have a third line in the plane, say $\mathbf{x} + T$, which is perpendicular to both L and M, and let $\mathbf{g}$ be a nonzero vector in T; note that T must be contained in W. It will follow that $\{\,\mathbf{e},\,\mathbf{f},\,\mathbf{g}\,\}$ is a set of nonzero mutually perpendicular vectors and hence is linearly independent. This is impossible; since W is 2–dimensional, every linearly independent subset of it contains at most two vectors. Therefore a third perpendicular cannot exist.∎

4. We use similar ideas to those of the preceding exercise. Clearly we can form another line $\mathbf{x} + T$ in this case, where T is spanned by $\mathbf{e} \times \mathbf{f}$. If we are given any line $\mathbf{x} + S$ perpendicular to L and M, then S is spanned by a vector $\mathbf{h}$ which is perpendicular to $\mathbf{e}$ and $\mathbf{f}$; since all such vectors are scalar multiples of the cross product, it follows that $\mathbf{x} + S$ must be the previously described line $\mathbf{x} + T$.■

5. Let $\mathbf{y}$ be the point where the two lines meet, let L be the line joining $\mathbf{y}$ and $\mathbf{y} + \mathbf{v}$, and let M be the line joining $\mathbf{y}$ and $\mathbf{y} + \mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors. Since the lines are assumed to be distinct, neither $\mathbf{v}$ nor $\mathbf{w}$ is a scalar multiple of the other. The points on L have the form $\mathbf{x}(t) = \mathbf{y} + t\mathbf{v}$ for some $t \in \mathbb{R}$. To find the perpendicular projection of this point on L, write $t\mathbf{v}$ as a sum of two vectors, one of which is a multiple of $\mathbf{w}$ and the other or which is perpendicular to $\mathbf{w}$. By the perpendicular projection formula the summand perpendicular to $\mathbf{w}$ is given by

$$\mathbf{z}(t) \quad = \quad t\mathbf{v} \; - \; \frac{\langle t\mathbf{v}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}$$

and the foot M_X of the perpendicular to L is given by

$$\mathbf{y} \; + \; \frac{\langle t\mathbf{v}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}$$

so that the distance from $\mathbf{x}(t)$ to M is just the length of $\mathbf{z}(t)$. It follows from the formulas that this distance is equal to $|t|\,|\mathbf{z}(t)|$; since $\mathbf{x}(1) = \mathbf{v}$ and $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, it follows that $\mathbf{z}(1) \neq \mathbf{0}$. Hence the distance from $\mathbf{x}(t)$ to M is $|t|$ times the positive real number $|\mathbf{z}(1)|$. Therefore the distance from $\mathbf{x}(t)$ to M is equal to $a > 0$ if and only if $t = \pm\, a/|\mathbf{z}(1)|$.■

III.2 : Basic theorems on triangles

1. Suppose that $\Delta ABC \cong \Delta DEF$. Then $d(A,B) = d(D,E)$ and

$$d(B,G) \quad = \quad \tfrac{1}{2}\, d(B,C) \quad = \quad \tfrac{1}{2}\, d(E,F) \quad = \quad d(E,H)$$

and since $\angle CBA = \angle GBA$ and $\angle FED = \angle HED$ we also have

$$|\angle GBA| \quad = \quad |\angle CBA| \quad = \quad |\angle FED| \quad = \quad |\angle HED| \; .$$

Therefore we have $\Delta GBA \cong \Delta HED$ by **SAS**.

Conversely, if $\Delta ABG \cong \Delta DEH$. Then as before we have $d(A,B) = d(D,E)$, and

$$d(B,C) \quad = \quad 2\, d(B,G) \quad = \quad 2\, d(E,H) \quad = \quad d(E,F)$$

and the reasoning in the previous paragraph yields

$$|\angle CBA| \quad = \quad |\angle GBA| \quad = \quad |\angle HED| \quad = \quad |\angle FED| \; .$$

Therefore we have $\Delta CBA \cong \Delta FED$ by **SAS**.■

2. Since $d(A,B) = d(A,C)$, $d(A,D) = d(A,D)$ and the bisection condition implies

$$|\angle DAB| \quad = \quad \tfrac{1}{2}\,|\angle CAB| \quad = \quad |\angle DAC|$$

it follows that $\Delta DAB \cong \Delta DAC$ by **SAS**. Therefore we also have $d(A,D) = d(B,D)$, so that D must be the midpoint of $[BC]$. Combining the latter with $d(A,B) = d(A,C)$, we see that AD is the perpendicular bisector of $[BC]$, so that $AD \perp BC$.■

3. By the Isosceles Triangle Theorem we have $|\angle PTS| = |\angle PST|$. Therefore by the Supplement Postulate for angle measure we have

$$|\angle PLT| \;=\; 180 - |\angle PTS| \;=\; 180 - |\angle PST| \;=\; |\angle PSR|$$

so that $\Delta PLT \cong \Delta PSR$ by **SAS** and the hypothesis $d(R,S) = d(L,T)$. The triangle congruence implies that $d(P,L) = d(P,R)$.

We are given the betweenness conditions $R*S*T$, $R*S*L$ and $R*T*L$, and from these we conclude that $L*T*S$ is also true. Combining $L*T*S$ and $R*S*T$ with the distance equations, we find that

$$d(L,S) \;=\; d(L,T) + d(T,S) \;=\; d(R,S) + d(S,T) \;=\; d(R,T)$$

and if we combine this with the previously obtained relations we see that $\Delta RTP \cong \Delta LSP$ by **SSS**.■

4. Since $C*B*D$ and $A*B*E$ hold, it follows that A and E lie on opposite sides of CD. Therefore we shall have $AC||DE$ if $|\angle ACD| = |\angle CDE|$; since $B \in (CD)$, the latter is equivalent to $|\angle ACB| = |\angle BDE|$.

To prove the final statement, note first that the common midpoint condition implies that $d(A,B) = d(B,E)$ and $d(C,B) = d(B,D)$. By the Vertical Angle Theorem we also have $|\angle ABC| = |\angle DBE|$, and therefore by **SAS** we have $\Delta ACB \cong \Delta DBE$. The desired equation $|\angle ACB| = |\angle BDE|$ is an immediate consequence of this.■

5. We shall follow the hint and first verify the betweenness relationships $A*C*E$ and $F*C*G$. First of all, $B*C*D$ and $C \in AE$ imply that B and D lie on opposite sides of AE. Next, $A*F*B$ and $D*G*E$ imply that B and F lie on the same side of AE and also that D and G lie on the same side of AE. This means that F and G must lie on opposite sides of AE, and since $C \in AE \cap FG$ this means that $F*C*G$ must be true. Furthermore, $A*F*B$, $D*G*E$ and $F*C*G$ imply that A and E lie on opposite sides of BC.

We can now use the Vertical Angle Theorem to conclude that $|\angle BCA| = |\angle DCE|$, and therefore $\Delta BCA \cong \Delta DCE$ by **SAS**. By the Alternate Interior Angle Theorem, we also know that AB is parallel to BE.

To conclude tha proof, we can now use the Alternate Interior Angle Theorem again to conclude that $|\angle FAC| = |\angle GEC|$, and another application of the Vertical Angle Theorem implies that $|\angle FCA| = |\angle GCE|$. Since we are given that $d(A,C) = d(C,E)$, it follows that $\Delta FAC \cong \Delta GCE$ by **ASA**.■

6. Since the angle sum of a triangle is 180 degrees, by the Isosceles Triangle Theorem and $\angle BAC = \angle DAE$ we have

$$|\angle ABC| \;=\; \tfrac{1}{2}(180 - |\angle BAC|) \;=\; \tfrac{1}{2}(180 - |\angle DAE|) \;=\; |\angle ADE|\,.$$

Therefore the Corresponding Angles criterion implies that $BC||DE$.■

7. Let α, β, γ and δ be the measures of $\angle BAC$, $\angle ABC$, $\angle ACB$, and $\angle ADB$. We know that $d = 130$, but for the time being it is simpler to file this away for future use.

By the theorem on angle sums of a triangle we have

$$\tfrac{1}{2}\alpha + \tfrac{1}{2}\beta + \delta \;=\; 180 \;=\; \alpha + \beta + \gamma$$

and if we subtract half the second equation from the first and afterwards multiply both sides by two we obtain

$$2\,\delta \;-\gamma \;\;=\;\; 180\;.$$

If we substitute $\delta = 130$ and solve for γ we find that $\gamma = 80$.■

8. By the Vertical Angle Theorem we have $|\angle ABD| = |\angle CBF|$, and since $|\angle ADB| = 90 = |\angle BCF|$, we may apply the "Third Angles Are Equal" theorem to conclude that $|\angle DAB| = |\angle BFC|$.■

9. If Y is an arbitrary point on L then $d(B,Y) = d(C,Y)$ because L is the perpendicular bisector of $[BC]$. It follows that $d(A,Y) + d(Y,B) = d(A,Y) + d(Y,C)$. The right hand side is minimized when Y is between A and C, and this happens precisely when Y is the point X where AC meets L; note that this point is between A and C because A and C lie on opposite sides of L. There is only one point $X \in L$ with these properties, so we know that $d(A,Y) + d(Y,B) = d(A,Y) + d(Y,C) > d(A,C)$ for all other points Y on the line L.■

10. The points X, Y, Z are not collinear because a line cannot intersect all three open sides of a triangle. Also, the betweenness hypotheses imply

$d(A,B) = d(A,X) + d(X,B)$,

$d(B,C) = d(B,Y) + d(Y,C)$, and

$d(A,C) = d(A,Z) + d(Z,C)$.

Finally, the strong form of the Triangle Inequality (for noncollinear triples) implies that

$d(X,Y) < d(B,X) + d(B,Y)$,

$d(Y,Z) < d(C,Y) + d(C,Z)$, and

$d(X,Z) < d(A,X) + d(A,Z)$.

If we add these we obtain

$$d(X,Y) \;+\; d(Y,Z) \;+\; d(X,Z) \;\;<\;\; d(B,X) \;+\; d(B,Y) \;+\; d(C,Y) + d(C,Z) \;+\; d(A,X) \;+\; d(A,Z)$$

and using the betweennes identities we see that the right hand side is equal to $d(A,B) + d(B,C) + d(A < C)$; thus we have shown the inequality stated in the exercise.■

11. We know that $D = \frac{1}{2}(A + B)$ and $E = \frac{1}{2}(A + C)$, so that $d(D,E)$ is equal to $|D - E| = \left|\frac{1}{2}(B - C)\right|$; since the latter is equal to $\frac{1}{2}|B - C|$, the conclusion of the exercise follows.■

12. Let E be the midpoint of $[AB]$. Then by the final result in Section I.4 we know that $D - E = \frac{1}{2}(C - B)$. Since $E \in (AB)$ and $AD \neq AB$ we know that $A,\ D,\ E$ are noncollinear, and thus by the Triangle Inequality for noncollinear points we have

$$d(A,D) \;\;<\;\; d(D,E) \;+\; d(A,E) \;\;=\;\; \tfrac{1}{2}\big(\,d(A,C) \;+\; d(A,B)\,\big)$$

which is the inequality stated in the exercise.■

13. Apply the theorem on angle sums of a triangle to the four triangles described in the hint to obtain the following equations:

$$|\angle CAB| \;+\; |\angle ABC| \;+\; |\angle BCA| \;\;=\;\; 180$$

$$\begin{aligned} |\angle XAB| + |\angle ABX| + |\angle BXA| &= 180 \\ |\angle CAX| + |\angle AXC| + |\angle XCA| &= 180 \\ |\angle CXB| + |\angle XBC| + |\angle BCX| &= 180 \end{aligned}$$

Adding the last three equations, we obtain

$$|\angle XAB| + |\angle ABX| + |\angle BXA| + |\angle CAX| + |\angle AXC| +$$

$$|\angle XCA| + |\angle CXB| + |\angle XBC| + |\angle BCX| = 540 .$$

Since X lies in the interior of ΔABC it lies in the interiors of all the angles $|\angle CAB|$, $|\angle ABC|$, $|\angle BCA|$ and therefore we have

$$\begin{aligned} |\angle CAB| &= |\angle CAX| + |\angle XAB| \\ |\angle ABC| &= |\angle ABX| + |\angle XBC| \\ |\angle BCA| &= |\angle BCX| + |\angle XCA| \end{aligned}$$

If we substitute this into the previous equation we obtain

$$|\angle CAB| + |\angle ABC| + |\angle BCA| + |\angle AXB| + |\angle AXC| + |\angle XBC| = 540$$

and if we now use $|\angle CAB|+|\angle ABC|+|\angle BCA| = 180$ and subtract 180 from both sides we obtain

$$|\angle AXB| + |\angle AXC| + |\angle XBC| = 360$$

which is the equation stated in the exercise.■

14. Let $x = d(A,B) = d(D,E)$. Then by the Pythagorean Theorem we have $d(E,F) = \sqrt{x^2 - d(D,F)^2}$ and $d(B,C) = \sqrt{x^2 - d(A,C)^2}$. If $d(E,F) < d(B,C)$, then the formulas in the preceding sentence imply $d(A,C) < d(D,F)$.■

15. By Exercise 12, we know that $d(A,C) < d(E,C) < d(B,C)$; since the larger angle is opposite the longer side, it follows that $|\angle CEA| < |\angle CAE|$. On the other hand, the Exterior Angle Theorem implies that $|\angle CEB| > |\angle CAE|$, so that $|\angle CEB| > |\angle CEA|$. Since we also have $|\angle CEB|+|\angle CEA| = 180$, it follows that $|\angle CEB| > 90 > |\angle CEA|$. Therefore $\angle CEA$ is an **ACUTE** angle.■

16. Suppose we are given numbers $a \leq b \leq c$; then these numbers are consistent with the strong Triangle Inequality if and only if $c < a + b$. So if this fails, then there is no triangle whose sides have the given lengths. In Section III.6 we show that, conversely, if the conditions in the first sentence hold, then one can realize the numbers as lengths of the sides of some triangle.

(a) Since $1 + 2 = 3$, these numbers do not satisfy the strong Triangle Inequality and hence cannot be the lengths of the sides of a triangle.■

(b) Since $4 < 5 < 6$ and $4 + 5 = 0 > 6$, these numbers are consistent with the Triangle Inequality.■

(c) Since $1 \leq 15 = 15$ and $1 + 15 > 15$, it follows that these numbers are consistent with the Triangle Inequality.■

(d) Since $1+5<8$, these numbers do not satisfy the strong Triangle Inequality and hence cannot be the lengths of the sides of a triangle.■

17. The strong Triangle Inequality implies that if there is a number x such that 10, 15, x are the lengths of the sides of a triangle, then $x+10>15$ and $x<10+15$. There is also the inequality $x+15>10$, but it is weaker than the first one. Therefore the conditions on x are that $5<x<25$.■

18. We know that $(n+1)^2 = n^2+(2n+1)$, so if we can write $(2n+1)=m^2$, it follows that $n^2+m^2=(n+1)^2$. Thus there is a right triangle whose sides have lengths n, m and $n+1$ by the Pythagorean Theorem.

Since there are infinitely many odd positive integers who are perfect squares, it follows that there are infinitely many choices of n and m such that the preceding holds.

To find all $n<100$ which satisfy this condition, it is necessary to find all $n<100$ such that $2n+1$ is a perfect square. In other words, we need to find all odd positive integers $m\geq 3$ such that $m^2<200$, and this is the set of all odd positive integers ≤ 13. We can retrieve n because it is equal to $\frac{1}{2}(m^2-1)$. The first few cases are then given as follows:

$$\begin{array}{l}
3^2 + 4^2 = 5^2 \\
5^2 + 12^2 = 13^2 \\
7^2 + 24^2 = 25^2 \\
9^2 + 40^2 = 41^2 \\
11^2 + 60^2 = 61^2 \\
13^2 + 84^2 = 85^2
\end{array}$$

Of course, this could be continued indefinitely.■

19. Apply the Law of Cosines to each triangle. Let $y=d(A,C)=d(A,D)$ and $z=d(A,B)$. Then we have

$$\begin{aligned}
d(B,C)^2 &= y^2 + z^2 - 2yz\cos|\angle CAB| \\
d(B,D)^2 &= y^2 + z^2 - 2yz\cos|\angle DAB|
\end{aligned}$$

It follows that $d(B,C)<d(B,D)$ if and only if $\cos|\angle DAB| > \cos|\angle CAB|$, and since the cosine function is strictly increasing between 0 and 180, the latter holds if and only if $|\angle DAB|>|\angle CAB|$, Therefore we have $d(B,C)<d(B,D)$ if and only if $|\angle DAB|>|\angle CAB|$, which is the conclusion of the Hinge Theorem.■

20. Before solving the problem, we should note that [1]–[3] are all theorems, while [4] is an immediate consequence of the Law of Cosines.

(a) By [3] we know that $\alpha=60^\circ$ so that the triangle is equiangular. By [2], this means that $a=b=c$. However, in the given data we have $a\neq b$, so the data are inconsistent and no triangle with the stated measurements can exist.■

(b) Property [1] is satisfied, but [4] is not because

$$a^2 + b^2 = 36 + 49 = 85 > 81 = c^2$$

should imply $\gamma<90^\circ$ and instead we have $\gamma=93^\circ$.■

21. First of all, note that **SSA** is satisfied for $\Delta BDC \leftrightarrow \Delta BDA$ because we are know that $d(B,D)=d(B,D)$, we are given that $d(B,C)=d(B,A)$, and we have $\angle BDC=\angle BDA$ because $C\in (DA$.

We know that ΔBCD is isosceles because $d(B,C) = d(C,D)$. Now $|\angle BCA| = 45^\circ$ because ΔABC is an isosceles triangle with a right angle at C, and this implies $|\angle CBD| = |\angle DBC|$; we shall call this number θ. Also, since $A * C * D$ is true we have

$$|\angle BCD| \quad = \quad 180^\circ \; - \; |\angle BCA| \quad = \quad 135^\circ \; .$$

and since the angle sum for ΔBCD is 180° it follows that $135 + 2\theta = 180$, so that $\theta = 22.5$.

Since C lies in the interior of $\angle ABD$ it follows that

$$|\angle ABD| \quad = \quad |\angle ABC| \; + \; |\angle CBD| \quad = \quad 90^\circ \; + \; 22.5^\circ \quad = \quad 112.5^\circ$$

while as before we have $|\angle CBD| = 22.5^\circ$. Finally we have $|\angle DAB| = |\angle CAB| = 45^\circ$, while our previous calculations show that $|\angle DCB| = 135^\circ$.■