SOLUTIONS TO EXERCISES FOR

MATHEMATICS 133 — Part 5

Winter 2009

NOTE ON ILLUSTRATIONS. Drawings for several of the solutions in this file are available in the following file:

`http://math.ucr.edu/~res/math133/math133solutions5figures.pdf`

III. Basic Euclidean concepts and theorems

III.3 : Convex polygons

1. We shall use the theorem stating that the line joining the midpoints of two sides of a triangle is parallel to the third side (see Section I.4). Applying this to ΔABD and ΔCBD, we conclude that $PS||BD$ and $QR||BD$. Therefore it follows that either $PS||QR$ or else $PS = QR$. Suppose that the latter is true; we know that PS and A lie on the same side of BD, while QR and C lie on the same side of BD. Thus $PS = QR$ implies that A and C lie on the same side of BD. However, this is impossible because we know that A and C lie on opposite sides of BD (the diagonal segments of a convex quadrilateral have a point in common). Therefore we have $PS||QR$.

A similar argument holds for PQ and RS. Both of these lines are parallel to AC by applying the triangle theorem to ΔABC and ΔADC, showing that $PQ||AC$ and $RS||AC$. We can now argue as in the previous paragraph that $PQ \neq RS$, so that the lines PQ and RS are parallel. It follows that P, Q, R, S form the vertices of a parallelogram.■

2. By the preceding result we know that P, Q, R, S form the vertices of a parallelogram, and it follows that (PR) and (QS) meet at their common midpoint.■

3. Following the hint, we shall use vector methods. The parallelogram condition implies that $C = B + D - A$ (see the exercises for Unit I), and the midpoint conditions imply that $E = \frac{1}{2}(A+B)$ and $F = \frac{1}{2}(C+D)$. To show that E, B, F, D form the vertices of a parallelogram, it will suffice to show that $F = B + D - E$.

If we substitute the expression $C = B + D - A$ in the midpoint equation for F, we find that $F = D + \frac{1}{2}(B-A)$, and if we substitute the expression for E in terms of A and B into $B+D-E$, we find that the latter is also equal to $D + \frac{1}{2}(B-A)$. Combining these equations, we find that $F = B + D - E$ as desired, so that the four points in the given order form the vertices of a parallelogram.

4. First of all, we know that C lies in the interior of $\angle DAB$. Next, by the Isosceles Triangle Theorem we know that $|\angle DAC| = |\angle DCA|$. Now $AB||CD$ and the Alternate Interior Angle Theorem imply that $|\angle DCA| = |\angle CAB|$, and thus we have $|\angle DAC| = |\angle CAB|$, which means that $[AC$ bisects $\angle DAB$.■

5. We know that $\angle ADE = \angle ADB$ and $\angle CBF = \angle CBD$. Since $AD||BC$, the Alternate Interior Angle Theorem implies that $|\angle ADE| = |\angle CBF|$. Since A, B, C, D form the vertices of a parallelogram, it follows that $d(A, D) = d(B, C)$; combining these observations with the assumption that $d(B, F) = d(D, E)$, we conclude that $\Delta ADE \cong \Delta CBF$. Therefore we have that $|\angle AED| = |\angle CFB|$, and by the Supplement Postulate for angle measurement we then also have

$$|\angle AEF| \;=\; 180 - |\angle AED| \;=\; 180 - |\angle CFB| \;=\; |\angle CFD| \,.$$

Now A and C lie on opposite sides of $EF = BD$, and if we combine this with the displayed equation and the Alternate Interior Angle Theorem we conclude that AE must be parallel to CF.■

6. Before proving this result, for the sake of completeness we include a verification that the diagonals (AC) and (BD) of a parallelogram $ABCD$ meet in their common midpoint, which we shall call E. The fastest say to do this is algebraically, using the fact that $C = B + D - A$ and then checking directly that $\frac{1}{2}(A + C) = \frac{1}{2}(B + D)$.

Suppose now that we have a parallelogram $ABCD$ which is a rhombus. Then $d(A, B) = d(C, B)$ and $d(A, D) = d(C, D)$ imply that BD is the perpendicular bisector of $[AC]$ and hence $AC \perp BD$.

Conversely, suppose that $AC \perp BD$. Since E is the midpoint of both $[AC]$ and $[BD]$, it follows that AC is the perpendicular bisector of $[BD]$ and BD is the perpendicular bisector of $[AC]$. The first conclusion implies that $d(B, A) = d(D, A)$, and since we have $d(A, B) = d(C, D)$ and $d(B, C) = d(A, D)$ for every parallelogram it follows that all four sides of $ABCD$ have the same length.■

7. Since ΔEAB is an equilateral triangle, we have $|\angle EAB| = |\angle EBA| = |\angle AEB| = 60$. The point E is assumed to lie in the interior of the square, so we then have

$$90 \;=\; |\angle DAB| \;=\; |\angle EAB| + |\angle EAD| \;=\; 60 + |\angle EAD|$$

$$90 \;=\; |\angle CBA| \;=\; |\angle EBA| + |\angle EBC| \;=\; 60 + |\angle EBC|$$

and therefore we have $|\angle EAD| = |\angle EBC| = 30$. Since the sides of a square have equal length, it follows that $\Delta EAD \cong \Delta EBC$ by **SAS**. This means that $|\angle AEB| = |\angle CEB|$ and $d(D, E) = d(C, E)$. The latter in turn implies $\angle EDC| = |\angle ECD|$.

In order to compute the measures of the angles in the preceding sentence, we need one more property of the figure. Since ΔABE is equilateral and $ABCD$ is a square, it follows that $d(A, D) = d(A, B) = d(A, E)$ and $d(B, C) = d(A, B) = d(B, E)$, so that ΔAED and ΔBEC are isosceles and hence $|\angle AED| =$ —∠ADE—*and*—∠BEC—$= |\angle BCE|$.

To simplify the algebra, let $x = |\angle EDC|$ and $y = |\angle EDA|$. The preceding observations then imply that $x + y = 90$ and $30 + 2y = 180$. If we solve these equations for x and y we obtain $y = 75$ and $x = 15$, and therefore it follows that $|\angle EDC| = |\angle ECD| = 15$.■

8. By the assumption in the exercise we know that (AC) and (BD) meet at some point E. Since we have $A * E * C$ and $B * E * D$, it follows from the theorems on order and separation that

C and D lie on the same side of AB,
A and B lie on the same side of CD,
B and C lie on the same side of AD, and
A and D lie on the same side of BC.

Therefore the four points A, B, C and D (taken in the alphabetical ordering) form the vertices of a convex quadrilateral.■

9. Follow the hint and use the conclusion of the preceding exercise. If the four points form the vertices of a convex quadrilateral (taken in the alphabetical ordering), then (AC) and (BD) have a point E in common by the proposition in the notes. The point E then lies in the interior of $\angle ABC$, and since we have $B * E * D$ it follows that the open ray $(BE = (BD$ also lies in the interior of $\angle ABC$. Furthermore, since $B * E * D$ holds and $E \in AC$, it follows that B and D lie on opposite sides of AC.

Conversely, suppose now that D lies in the interior of $\angle ABC$ and D and B lie on opposite sides of AC. The first of these implies that the open ray $(BD$ meets (AC) in some point E, and the second implies that (BD) meets AC in some point F. Since both E and F lie on the intersection of the (distinct!) lines AC and BD and these lines have at most one point in common, it follows that $E = F$. Finally, since $E \in (AC)$ and $F \in (BD)$, it follows that (AC) and (BD) have a point in common, which must be $E - F$.■

10. By the preceding exercise the points form the vertices of a convex quadrilateral (taken in the alphabetical ordering) if and only if D lies in the interior of $\angle ABC$ and B and D lie on opposite sides of AC. The first of these implies x and z are positive, and the second implies that y is negative.

Conversely, suppose we have the conditions on the barycentric coordinates in the preceding sentence. Since y is negative, it follows that B and D lie on opposite sides of AC, and since the other two barycentric coordinates are positive it follows that D lies in the interior of $\angle ABC$.■

11. The assumptions are equivalent to saying that $C - D$ is a nonzero multiple of $B - A$, so write $C - D = k(B - A)$, where $k \neq 0$. We then have

$$D \quad = \quad kA \ - \ kB \ + \ C$$

and since the coefficients on the right hand side add up to 1 they give the barycentric coordinates of D with respect to A, B and C. By the preceding exercise, we know that the four points form the vertices of a convex quadrilateral (taken in the alphabetical ordering) if and only if k is positive.■

12. By the preceding exercise we know that $C - D = k(B - A)$, where $k > 0$. As before we have $D = kA - kB + C$, and the conditions $y = d(A, B)$ and $x = d(C, D)$ also imply $x = ky$. The midpoints G and H of $[AD]$ and $[BC]$ are then given by $H = \frac{1}{2}(B + C)$ and $G = \frac{(1+k)}{2}A - \frac{k}{2}B + \frac{1}{2}C$. It follows that

$$H \ - \ G \quad = \quad \tfrac{(1+k)}{2}\big(B \ - \ A\big)$$

so that GH is parallel to AB and CD, and furthermore we have

$$d(G, H) \quad = \quad \tfrac{(1+k)}{2} \cdot d(A, B) = \tfrac{(1+k)}{2} \cdot y \quad = \quad \tfrac{(y+ky)}{2} \quad = \quad \tfrac{(x+y)}{2}$$

as stated in the exercise.

To prove the remaining parts of the exercises, it suffices to show that the midpoints of $[AC]$ and $[BD]$ lie on the line GH. Let S and T denote these respective midpoints. Then we know that GS is parallel to AB since it joins the midpoints of two sides of ΔABD, and by Playfair's Postulate it follows that GS must be the same as GH, which is also a line through G which is parallel to AB. Similarly, the lines HG and HT are parallel to AB, and therefore $GH = HG = HT$. Thus we have shown that both S and T lie on GH. Note that $S \neq T$ unless the quadrilateral is a parallelogram (a standard result in plane geometry states that if the diagonals bisect each other, then the quadrilateral is a parallelogram).■

13. First of all, we have $A * E * B$ because $E \in (AB$ and $d(A,E) = x < y = d(A,B)$. Choose $k > 0$ so that $C - D = k(B - A)$ and hence $x = ky$. If $m > 0$ is chosen such that $E - A = m(B - A)$, then we have

$$m \cdot d(A,B) \quad = \quad d(A,E) \quad = \quad d(C,D) \quad = \quad k \cdot d(A,B$$

so that $m = k$. Therefore we also have $E - A = k(B - A) = C - D$. Since $A,\ C,\ D$ are noncollinear, this means that $A,\ E,\ C,\ D$ (in that order) form the vertices of a parallelogram. In particular, we know that AD is parallel to CE.■

14. By the preceding exercise we know that $A,\ E,\ C,\ D$ (in that order) form the vertices of a parallelogram. Since consecutive angles of a parallelogram are supplementary, it follows that $|\angle DAB| + |\angle AEC| = 180$. However, by the preceding exercise we also know that $A * E * C$ and thus by the Supplement Postulate we also know that $|\angle AEC| + |\angle CEB| = 180$. Combining these, we see that $|\angle DAB| = |\angle CEB|$ in all cases. Furthermore, since the opposite sides of a parallelogram have equal length, we also know that $d(A,D) = d(E,C)$. We shall use this fact repeatedly in proving the equivalence of the three conditions in the exercise.

Proof that (1) $\Longrightarrow$ (2). In this case we are given $d(A,D) = d(B,C)$. By the discussion above we have $d(B,C) = d(A,D) = d(E,C)$. Therefore the Isosceles Triangle Theorem implies that $|\angle CEB| = |\angle CBE|$, and since $\angle CBE = \angle CBA$ it follows that $|\angle CEB| = |\angle CBA|$. On the other hand, by the general discussion we have $|\angle DAB| = |\angle CEB|$, and therefore it follows that $|\angle DAB| = |\angle CBA|$ as required.■

Proof that (2) $\Longrightarrow$ (3). Since consecutive angles in a parallelogram are supplementary, it follows that $|\angle ADC| = 180 - |\angle DAB|$. Since $|\angle DAB| = |\angle CBA|$, it will suffice to show that $|\angle BCD| = 180 - |\angle CBA|$. There are several ways to do this, but the fastest might be to switches the roles of A and C with those of B and D in the preceding discussion; this can be done because the hypothesis does not change if one switches symbols in this fashion. — An alternative approach (which we shall only sketch) is to check that E lies in the interior of $\angle BCD$ and that $|\angle ECD| = |\angle DAB|$ (opposite angles of a parallelogram have equal measure), so that $|\angle BCD| = |\angle ECD| + |\angle ECB| = |\angle ECD| + |\angle DAB| = |\angle ECD| + |\angle CEB|$. Since $|\angle ECD| + |\angle CEB| + |\angle ABC| = 180$ it follows that $|\angle BCD| = 180 - |\angle CBA|$ as required.■

Proof that (3) $\Longrightarrow$ (1). One way of doing this is to show that (3) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1). Each of these can be done by reversing the steps in the preceding parts of the exercise.■

15. Let X and Y be the midpoints of $[AB]$ and $[CD]$ respectively. By construction we then have $d(A,X) = d(X,B)$ and $d(C,Y) - d(Y,C)$

By the preceding exercise we know that $d(A,D) = d(B,C)$ and also that $|\angle DAX| = |\angle CBX|$ as well as $|\angle ADY| = |\angle BCY|$. It follows that $\Delta DAX \cong \Delta CBX$ and $\Delta ADY \cong \Delta BCY$. These congruences imply $d(X,D) = d(X,C)$ and $d(Y,A) = d(Y,B)$. The first of these implies that XY is the perpendicular bisector of $[CD]$, and the second implies that XY is the perpendicular bisector of $[AB]$. Therefore the line XY is perpendicular to both AB and CD.■

16. We need to find s and t such that $0 < s,t < 1$ and $sC + (1-s)A = tD + (1-t)B$. If we substitute the coordinate expressions for the four points $A,\ B,\ C,\ D$ we obtain the following equations for the coordinates:

$$\begin{aligned} \tfrac{1}{2}sy \;+\; (1-s)(-\tfrac{1}{2}x) \quad &= \quad -\tfrac{1}{2}ty \;+\; (1-t)(\tfrac{1}{2}x) \\ sh \quad &= \quad th \end{aligned}$$

The second equation implies $s = t$, and if we substitute this into the first equation we obtain

$$\tfrac{1}{2}sy \ + \ (1-s)(-\tfrac{1}{2}x) \ = \ -\tfrac{1}{2}sy \ + \ (1-s)(\tfrac{1}{2}x)$$

which means that the right hand side is the negative of the left hand side and hence both are equal to zero. Therefore the equations imply $sy = (1-s)x$ and $k = sh$.

We can solve this for s to obtain $s = x/(x+y)$, and if we substitute this and $k = sh$ into $k/(h-k)$, it follows that the latter is equal to x/y.■

17. (a) Following the hint, one first notes that

$$d(A,B) \ = \ d(B,C) \ = \ d(C,D) \ = \ d(D,A) \ = \ \sqrt{p^2+q^2}$$

and then one notes that $A - B \ = \ (p,-q) \ = \ D - C$. Therefore one has

$$AB \ = \ A \ + \ \mathbf{R}\cdot(p,-q) \ , \qquad CD \ = \ C \ + \ \mathbf{R}\cdot(p,-q)$$

so that the lines AB and CD are either parallel or identical. The latter is impossible because it would imply that A, B, C would be collinear. Since the defining equation for AB is $f(x,y) \ = \ qx + py - pq \ = \ 0$ and $f(C) \ = \ q(-p) + p\cdot 0 - pq \ = \ -2pq \ < \ 0$, we know that A, B, C cannot be collinear, and hence $AB||CD$.

If we interchange the roles of B and D in the preceding argument, we obtain the analogous conclusion that $AD||BC$. Combining these with the observations in the first paragraph, we conclude that A, B, C, D form the vertices of a parallelogram, and (by the first sentence of that paragraph) this parallelogram is a rhombus.■

(b) As suggested in the hint, let T be the orthogonal linear transformation on $\mathbf{R}^2$ defined by $T(x,y) = (x,-y)$; geometrically and physically, the mapping T corresponds to reflection about the x-axis. By definition the map T sends A and C to themselves, and it interchanges B and D. Since a set of points in $\mathbf{R}^2$ is collinear if and only if its image is collinear, it follows that T must interchange the lines AB and AD, and it must also interchange the lines BC and BD.

Suppose now that $F \in AB$ and $G \in CD$ are such that $FG \perp AB$ and $FG \perp CD$. Since T preserves angle measurements, it follows that the line $T(F)\,T(G)$ is perpendicular to both AD and $CB = BC$. Furthermore, since T is distance-preserving, it follows that

$$d(F,G) \ = \ d\,(\,T(F),\,T(G)\,) \ .$$

By construction, the left hand side is equal to the distance between the parallel lines AB and CD, whild the right hand side of the equation is equal to the distance between the parallel lines AD and BC.■

18. As suggested by the drawing, we have $b + 2c = a$ and $b = c\sqrt{2}$. Therefore we have

$$b \ = \ a \ - \ 2\,c \ + \ a \ - \ \frac{2b}{\sqrt{2}} \ = \ a \ - \ b\sqrt{2}$$

, which means that $a = (1+\sqrt{2})b$; if we solve for b and put the result into simplified radical form, we find that $b = \left(\sqrt{2}-1\right)\,a$.■

19. By hypothesis we have $A * F * B$ and $C * E * D$, which mean that E, F, A, and D all lie on the same side of BC. For the same reasons, it also follows that E, F, B, and C all lie on the same side of AD. These imply that the closed segment (EF) is contained on the side of BC containing A and D, and it is also contained on the side of AD containing B and C. Next $C * E * D$ and $A * F * B$ imply first that E is on the same side of AB as C and D, and furthermore this side contains the open segment (EF). Likewise, the order conditions imply that F is on the same side of CD as A and B, and furthermore this side contains the open segment (EF). Now the interior of the quaerilateral is the intersection of the four half-planes which were discussed above, and therefore we have verified all four conditions needed to show that (EF) lies in the interior of the convex quadrilateral.

20. Suppose that L meets (AC) in some point X, and consider ΔABC. Then by Pasch's Theorem either L contains C or else L meets one of the sides (BC) or (AC). In all cases except the last one, the conclusion of the exercise follows immediately.

On the other hand, if L contains a point of (AC), then we can consider ΔACD. By Pasch's Theorem, we know that either L contains D or else L meets one of the sides (CD) or (AD). Thus in the last case of the preceding paragraph the conclusion of the exercise also holds.■

21. Suppose that A_3 and A_4 lie on opposite sides of A_1A_2. then there is a point $X \in A_1A_2 \cap (A_3A_4)$. Since no three of the original four points are collinear we know that $X \neq A_1, A_2$; therefore exactly one of the points X, A_1, A_2 is between the other two. Now if $A_1 * A_2 * X$, then it will follow that A_1 and A_2 are on the same side of A_3A_4, and likewise if $A_2 * A_1 * X$. Suppose now that we have $A_1 * X * A_2$. Then it follows that A_1 and X are on the same side of A_2A_3, and since $A_3 * X * A_4$ holds we also know that A_4 and X are on the same side of A_2A_3. Combining these observations, we conclude that A_1 and A_2 are on the same side of A_2A_3, and hence the conclusion of the exercise holds in all possible cases.■

22. The distance conditions imply that both A and C lie on the perpendicular bisector of $[BD]$, and since they are distinct points it follows that AC must be the perpendicular bisector of $[BD]$. We can now apply **SSS** to ΔABC and ΔADC to conclude that $\Delta ABC \cong \Delta ADC$, and from this it follows that $|\angle ABC| = |\angle ADC|$.■

III.4 : Concurrence theorems

1. Let f denote the function under consideration, so that

$$f(X) \quad = \quad |X-A|^2 + |X-B|^2 + |X-C|^2 \quad = \quad 3|X|^2 - 2\langle X,\, A+B+C\rangle + |A|^2 + |B|^2 + |C|^2 \ .$$

For the time being let us assume this function has an absolute minimum value somewhere. Given a point P in the plane, write it as $P = (P_1, p_2)$. Then the minimum occurs when the partial derivatives of f vanish. This corresponds to the system of equations $6x_i - 2(a_i + b_i + c_i) = 0$. The solution to this system is given by $x_i = \frac{1}{3}$, so that X is simply the barycenter of $A,\ B$ and C.

How do we know that this is a minimum? If we apply the second derivative test we find that the mised second partial derivative $f_{1,2} = f_{2,1}$ is identically zero and that $f_{1,1}$ and $f_{1,2}$ are both equal to 6 everywhere. Therefore the second partial derivatives satisfy $f_{1,1}f_{2,2} - f_{1,2}^2 \ > \ 0$, and by standard results in multivariable calculus this means that f has a relative minimum at the barycenter Y.

Why is this an **absolute** minimum? Here is one simple way to check this. The function f has an absolute minimum over every disk defined by an inequality of the form $x_1^2 + x_2^2 \leq r^2$. We claim that if r is sufficiently large then this minimum value cannot occur on the boundary, and since it occurs at an interior point it must occur at some point where the first partial derivatives vanish.

We have seen that the barycenter is the only such point, so this is where there must be an absolute minimum.

Recall that the Triangle Inequality implies that $|\mathbf{u}-\mathbf{v}| \geq \big||\mathbf{u}|-|\mathbf{v}|\big|$ for all vectors $\mathbf{u}$ and $\mathbf{v}$. We shall use this fact in the next paragraph.

Let $z_0 \geq 0$ be the value of f at the barycenter Y, and choose r so large that $r > |Y|$ and $|A| + \sqrt{z_0}$. Then we have

$$|X-A|^2 \quad \geq \quad \big(|X|-|A|\big)^2 \quad > \quad z_0$$

and since $f(X) \geq |X-A|^2$ it follows that the value of $f(X)$ on the circle of radius r is always greater than $z_0 = f(Y)$. Since $|Y| < r$, it follows that the absolute minimum cannot occur on the boundary, so the argument of the preceding paragraph shows that the absolute minimum for this value of r must be realized at the barycenter Y.■

2. We have three isosceles triangles ΔVAB, ΔVBC and ΔVAC, and from these we conclude that $|\angle VAB| = |\angle VBA|$, $|\angle VBC| = |\angle VCB|$ and $|\angle VAC| = |\angle VCA|$. Since V lies in the interior of ΔABC it lies in the interior of all three vertex angles and hence we have

$$\begin{array}{rcl} |\angle CAB| & = & |\angle VAC| + |\angle VAB| \\ |\angle ABC| & = & |\angle VBA| + |\angle VBC| \\ |\angle BCA| & = & |\angle VCB| + |\angle VCA| \end{array}$$

Since $|\angle CAB| + |\angle ABC| + |\angle BCA| = 180$, it follows that

$$180 \quad = \quad |\angle VAC| + |\angle VAB| + |\angle VBA| + |\angle VBC| + |\angle VCB| + |\angle VCA| \quad =$$

$$2\big(\,|\angle VBC| + |\angle VAB| + |\angle VAC|\,\big)$$

and hence $|\angle VBC| + |\angle VAB| + |\angle VAC| = 90$. Now each angle measure is positive, and by the preceding equations we have

$$\begin{array}{rcl} |\angle CAB| & = & 90 - |\angle VBC| \\ |\angle ABC| & = & 90 - |\angle VAC| \\ |\angle BCA| & = & 90 - |\angle VAB| \end{array}$$

and therefore each of the angle measures on the left hand side is less than 90, so that all the vertex angles of the triangle are acute.■

3. Follow the hint. If E is the midpoint of $[BC]$ then VE is parallel to AB and hence VE is perpendicular to BC. It follows that VE is the perpendicular bisector of $[BC]$ and hence $d(V,B) = d(V,C)$, and since the latter is equal to $d(V,A)$ it follows that V is equidistant from all three vertices.■

4. We shall use some of the observations that were made in the solution to Exercise 2 of this section. In particular, one of the observations (mentioned in the hint) is that

$$|\angle VAC| \quad = \quad 90 \;-\; |\angle BAC| \quad = \quad 90 \;-\; \beta.$$

Since the hypotheses imply that $d(V,A) = R$ and $d(A,D) = \frac{1}{2}d(A,C) = b$, we may use the standard trigonometric formula in Theorem III.2.9A to conclude that

$$\cos\,(90 \;-\; \beta) \quad = \quad \cos|\angle VAC| \quad = \quad \frac{b}{2R}$$

and the formula in the exercise follows from this and the fact that sin β equals cos $(90 - \beta)$.■

5. First of all, we translate the problem into more mathematical terms. We are given perpendicular lines X (Queen's Road) and Y (King's Road), and we are also given four points E, M, P, S corresponding to the elm, maple, pine and spruce trees, with each of these points on either X or Y. The positions of all the points except M are fixed in the problem, and all we know about M is that it satisfies certain constraints. We are also told that the lines EP and MS meet at some point A, the lines SP and EM meet at some point B, and the line AB meets Y at some point T where the treasure is located. — The point of the exercise is to prove that T is the same for all choices of M consistent with the conditions in the problem.

As suggested in the hint, we shall begin by choosing a coordinate system in which the two main roads are the coordinate axes.[1] Therefore we may assume that $E = (-4, 0)$, $P = (0, 3)$, $S = (2, 0)$, and $M = (0, k)$ where $k > 3$. Standard results in analytic geometry imply that the equations of the four lines EP, MS, SP and EM are given as follows:

$$\begin{array}{rrcl} EP: & -3x + 4y - 12 & - & 0 \\ MS: & kx + 2y - 2k & = & 0 \\ SP: & 3x + 2y - 6 & = & 0 \\ EM: & -kx + +4y - 4k & = & 0 \end{array}$$

Two more applications of standard methods in analytic geometry then yield the coordinates for the intersection points.

$$A \in EP \cap MS \qquad \text{has coordinates} \qquad \left(\frac{8k - 24}{4k + 6}, \frac{18k}{4k + 6} \right)$$

$$B \in SP \cap ME \qquad \text{has coordinates} \qquad \left(\frac{24 - 8k}{2k + 12}, \frac{18k}{2k + 12} \right)$$

We now want the point where AB meets the x-axis. In fact, it is straightforward to derive the following:

FORMULA. *Suppose we are given two points* $\mathbf{a}$ *and* $\mathbf{b}$ *in* $\mathbf{R}^2$ *such that the line* $\mathbf{ab}$ *meets the x-axis. For* $\mathbf{x} = \mathbf{a}$ *or* $\mathbf{b}$, *write* $\mathbf{x}$ *in coordinates as* (x_1, x_2), *and let* $(\mathbf{a} : \mathbf{b})$ *denote the* 2×2 *matrix whose rows are given by* $\mathbf{a}$ *and* $\mathbf{b}$ *in that order. If* $\mathbf{q}$ *is the point where* $\mathbf{ab}$ *meets the x-axis, then the first coordinate of* $\mathbf{q}$ *is given by the following expression:*

$$\frac{\det(\mathbf{a} : \mathbf{b})}{b_2 - a_2}$$

We note that the denominator is nonzero, for if $b_2 = a_2$ then $\mathbf{ab}$ is parallel to the x-axis; of course, the second coordinate of $\mathbf{q}$ must be zero since this point lies on the x-axis.

[1] This is an informal way of proceeding, but it can be justified formally as follows: If we can prove that T is the same for all possible choices of M when X and Y are the coordinate axes, then the result is true in general. To see this, let G be a Galilean transformation which takes the coordinate axes to X and Y respectively. Then the solution to the problem in the special case implies that $G^{-1}(T)$ is the same for all possible choices of $G^{-1}(M)$, and therefore it follows that T is the same for all possible choices of M because G preserves lines (and intersections ot lines), distances, and angle measurement.

If we use this formula to compute the first coordinate of the point T, we find that this coordinate is equal to 8 for all admissible choices of k. Thus one can find the point T without knowing the explicit value for k.

Comments. It is somewhat remarkable that this fairly difficult problem was taken from an old high school geometry text (E. E. Moïse and F. L. Downs, *Geometry*, Addison-Wesley, 1964). — It is probably not at all clear how or why anyone came up with such a problem, but in fact it is related to the Theorem of Desargues and its dual result, both of which are discussed in Unit IV of the course notes (as noted elsewhere, Unit IV will not be covered during this quarter). Further details appear in the following online file:

`http://math.ucr.edu/~res/progeom/treasure-problem.pdf`

6. If V is the circumcenter of $\Delta\,ABC$, then one can find V by solving the system of equations

$$|V - A|^2 \;=\; |V - B|^2 \;=\; |V - C|^2\;.$$

If we let $V = (x, y)$ and substitute for A, B, C, then we obtain the equations $x + y = 6$ and $6x - 2y = 14$. The solution to this system of equations is $x = \frac{13}{4}$ and $y = \frac{11}{4}$.■

7. The equation of the line M which is perpendicular to L (with equation $y = 2x + 2$ and passes through $(1, 0)$ must have the form $y = -\frac{1}{2}x + C$ for some constant C. Since the line passes through $(1, 0)$ it follows that the constant must be $\frac{1}{2}$. Now the orthocenter is is a point which lies on all three altitudes, so it is enough to find a point where two of the altitudes meet. Since the altitude from $(0, 2)$ is just the y-axis, the point in question turns out to be $(0, \frac{1}{2})$.■

Alternate approach. Another way to find the orthocenter of a triangle $\Delta\,ABC$ is to find the circumcenter of the associated triangle $\Delta\,DEF$, where $D = C + B - A$, $E = A + C - B$, and $F = A + B - C$; recall that DE is the unique line through C which is parallel to AB, while EF is the unique line through A which is parallel to BC and DF is the unique line through B which is parallel to AC. — For the specific example in this exercise, we may take $A = (0, 2)$, $B = (-1, 0)$ and $C = (1, 0)$, so that $D = (0, -2)$, $E = (2, 2)$ and $F = (-2, 2)$. The equations for the circumcenter of $\Delta\,DEF$ reduce to $8x = 0$ and $4x + 8y = 4$, so that $x = 0$ and $y = \frac{1}{2}$. This is the same conclusion that was obtained in the preceding paragraph.■

8. The statement of the problem notes that there is a 60 degree angle at A and a 90 degree angle at B; it follows that there is a 30 degree angle at C. Following the hint, we see that the angle bisector for the vertex angle at B satisfies the equation $y = x$. Let $D \in (BC)$ be the point where the bisector of $\angle BAC$. Since the measure of the bisector of $\angle DAC$ is $30°$, the slope of the line containing this bisector is equal to $\tan 150° = -\tan 30° = -\frac{1}{3}\sqrt{3}$, and it follows that the line containing the bisector has equation

$$y \;=\; \frac{\sqrt{3}}{3}\,(1 \;-\; x)\;.$$

The point where this line meets the bisector of $\angle ABC$ satisfies the equations

$$x \;=\; y \;=\; \frac{\sqrt{3}}{3}\,(1 \;-\; x)$$

and the solution of this system, which gives the incenter, satisfies

$$y \;=\; x \;=\; \frac{\sqrt{3}}{3 + \sqrt{3}} \;=\; \frac{3\sqrt{3} - 3}{6}\;.■$$

9. In coordinates, the equations defining the circumradius are $x^2+y^2 = (x-3)^2+(y-4)^2 = (x-6)^2+y^2$. If we subtract x^2+y^2 from all sides of this equation we obtain the system of linear equations $0 = 25-6x-8y = 36-12x$. Solving these, we obtain $x=3$ and $y=7/8$.

The circumradius is equal to

$$\sqrt{3^2+\frac{7}{8}^2} \quad = \quad \sqrt{\frac{24^2+7^2}{8^2}} \quad = \quad \sqrt{\frac{25^2}{8^2}}$$

and hence the circumradius is equal to $25/8 = 3\frac{1}{8}$.■

10. We shall make the assumptions stated in the problem, so that $A=(0,h)$ for some $h\neq 0$ and B and C are $(\pm\sqrt{3}h,0)$; we might as well assume that B and C correspond to the $-$ and $+$ choices respectively.

For the circumcenter question, we need to find the point where the perpendicular bisector of $[AC]$ meets the y-axis (the latter is the perpendicular bisector of $[BC]$). Now the slope of AC is $-1/\sqrt{3}$ and the midpoint of $[AB]$ is $(\frac{1}{2}h\sqrt{3},\frac{1}{2}h)$, so the perpendicular bisector is the line which goes through this midpoint and has slope $\sqrt{3}$. Therefore the perpendicular bisector of $[AB]$ satisfies the equation

$$y \quad = \quad \sqrt{3}\left(x-\tfrac{1}{2}\sqrt{3}h\right) \ .$$

The point at which this line meets the y-axis is given by setting $x=0$, and hence it has coordinates $(0,-\frac{1}{6}h)$. Since this point's y-coordinate has the opposide sign as the y-coordinate of A, it follows that the circumcenter and A lie on opposite sides of the x-axis, which is BC. Therefore the circumcenter does not lie in the interior of ΔABC; furthermore, if the circumcenter lies on ΔABC, then it must be the midpoint of $[BC]$ and hence it is the point $(0,0)$. It follows that the circumcenter must lie in the exterior of ΔABC.

For the incenter question, we need to find the point where the y-axis meets the (perpendicular) altitude from B to AC. In this case we know that the slope is again $\sqrt{3}$ but now the line goes through the point $(-h\sqrt{3},0)$. The equation of this line is given by

$$y \quad = \quad \sqrt{3}\left(x+h\sqrt{3}\right)$$

and it meets the y-axis at the point $(0,3h)$. If we let J be this point (which is the incenter) and set $Q=(0,0)$, then it follows that $Q*A*J$. This means that Q and J lie on opposite sides of both AB and AC. Since Q is the midpoint of $[BC]$ it follows that $B*Q*C$ so that Q lies on the same side of AB as C and also on the same side of AC as B. Therefore J lies on the opposite side of AB as C, and likewise with B and C switched. It follows that the incenter lies int the exterior of ΔABC.■

11. Follow the hints, and choose $\mathbf{u}$, $\mathbf{v}$, $\mathbf{p}$, $\mathbf{q}$, $\mathbf{x}$, $\mathbf{y}$ as indicated. The angles $\angle\mathbf{pbq}$ and $angle\mathbf{abc}$ are identical because $[\mathbf{bp} = [\mathbf{ba}$ and $[\mathbf{bq} = [\mathbf{bc}$. Also, $\Delta\mathbf{bpq}$ is an isosceles triangle with $d(\mathbf{b},\mathbf{p}) = d(\mathbf{b},\mathbf{q})$, and by construction $\mathbf{x}$ is the midpoint of $[\mathbf{p}\,\mathbf{q}]$. Therefore $\Delta\mathbf{bpx}\cong\Delta\mathbf{bqx}$ by **SSS**,

12. In this case $\mathbf{u} = (\frac{3}{5},\frac{4}{5})$ and $\mathbf{v} = (2/\sqrt{5},1/\sqrt{5})$, so that the point on the bisector is

$$\mathbf{x} \quad = \quad \frac{1}{2}\left(\frac{3}{5}+\frac{2}{\sqrt{5}},\,\frac{4}{5}+\frac{1}{\sqrt{5}}\right) \quad =$$

$$\frac{1}{10}\left(3+2\sqrt{5},\, 4+\sqrt{5}\right) .$$

If $\mathbf{x} = (u, v)$ with $u \neq 0$, then the slope of the line $\mathbf{bx} = \mathbf{0x}$ is just the quotient v/u, and thus in this case the slope of the bisector is just

$$\frac{3+2\sqrt{5}}{4+\sqrt{5}} \quad = \quad \frac{2+5\sqrt{5}}{11}$$

where the latter is obtained by multiplying the numerator and denominator of the left hand side by $4 - \sqrt{5}$.■