SOLUTIONS TO EXERCISES FOR

MATHEMATICS 133 — Part 6

Winter 2009

NOTE ON ILLUSTRATIONS. Drawings for several of the solutions in this file are available in the following file:

`http://math.ucr.edu/∼res/math133/math133solutions6figures.pdf`

III. Basic Euclidean concepts and theorems

III.5 : Similarity

1. Write $\mathbf{T})(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ as usual. The condition in the exercises is that if the vectors $\mathbf{y} - \mathbf{x}$ and $\mathbf{z} - \mathbf{x}$ are perpendicular, then the same is true for $\mathbf{T}(\mathbf{y}) - \mathbf{T}(\mathbf{x})$ and $\mathbf{T}(\mathbf{z}) - \mathbf{T}(\mathbf{x})$. We then have

$$\mathbf{T}(\mathbf{y}) \; - \; \mathbf{T}(\mathbf{x}) \;\; = \;\; [A\mathbf{y} + \mathbf{b}] \; - \; [A\mathbf{x} + \mathbf{b}] \;\; =$$

$$A[\mathbf{y}] \; - \; A[\mathbf{x}] \;\; = \;\; A[\mathbf{y} - \mathbf{x}]$$

and $\mathbf{T}(\mathbf{z}) - \mathbf{T}(\mathbf{x}) = A[\mathbf{z} - \mathbf{x}]$, so the condition on $\mathbf{T}$ is equivalent to saying that A takes perpendicular vectors to perpendicular vectors.

The columns of A are the images of the standard unit vectors $\mathbf{e}_i$ under the linear transformation corresponding to left multiplication by A, and therefore the inner product of $A\mathbf{e}_i$ and $A\mathbf{e}_j$ is just the (i, j) entry of the matrix ${}^{\mathbf{T}}\!A\, A$. Since we know that A takes perpendicular vectors to perpendicular vectors, we know that the off-diagonal entries of this product are zero; since A is invertible, we also know that the diagonal entries $d_{i,i}$ must be positive real numbers.

We claim that all the entries $d_{i,i}$ are equal. Suppose that $i \neq j$ and consider the perpendicular vectors $\mathbf{e}_i \pm \mathbf{e}_j$. By the basic condition on A we know that $A\mathbf{e}_i + A\mathbf{e}_j$ is perpendicular to $A\mathbf{e}_i - A\mathbf{e}_j$, and if we expand this inner product we find it is equal to $d_{i,i} - d_{j,j}$, so it follows that the latter expression is equal to zero. Since i and j were arbitrary, it follows that all the diagonal entries of the matrix ${}^{\mathbf{T}}\!A\, A$ are equal to some fixed positive real number d.

If we take $C = \left(\sqrt{d}\right)^{-1} A$, then it follows that ${}^{\mathbf{T}}\!C\, C$ is the identity matrix, which implies that C is orthogonal and $\mathbf{T})(\mathbf{x}) = \sqrt{d}\, C\mathbf{x} + \mathbf{b}$ and hence $\mathbf{T}$ is a similarity transformation.■

2. There is a unique vector $\mathbf{z}$ such that $\mathbf{T}(\mathbf{z}) = \mathbf{z}$ if and only if there is a unique vector $\mathbf{z}$ such that $A\mathbf{z} + \mathbf{b} = \mathbf{z}$, which in turn is equivalent to the existence of a unique $\mathbf{z}$ such that $(kA - I)\mathbf{z} = -\mathbf{b}$. The latter is true if and only if $kA - I$ is invertible, and this is true if and only if there is no nonzero vector $\mathbf{v}$ such that $(kA - I)\mathbf{v} = \mathbf{0}$. Suppose that the last condition is false, and let $\mathbf{v} \neq \mathbf{0}$ be such a vector. Then we have $kA\mathbf{v} = \mathbf{v}$.

It follows that $|\mathbf{v}| = |kA\mathbf{v}|$. Since k is a positive constant not equal to 1, the right hand side is equal to $k|A\mathbf{v}|$, and since the orthogonal matrix A is length-preserving it follows that $|A\mathbf{v}| = \mathbf{v}$.

Combining these observations, we see that $0 \neq |\mathbf{v}| = k|\mathbf{v}|$. However, this is impossible since $k \neq 1$, and therefore we have a contradiction. The source of the problem is our assumption that there is a nonzero vector $\mathbf{v}$ such that $(kA - I)\mathbf{v} = \mathbf{0}$, and hence no such vector can exist. By the arguments of the preceding paragraph, this implies the conclusion of the exercise.■

3. The Angle Bisector Theorem implies that

$$\frac{d(A,D)}{d(B,D)} = \frac{d(A,C)}{d(B,C)} = \frac{3}{4}$$

and since $d(A,D) + d(D,B) = d(A,B) = 5$ it follows that $x = d(A,D)$ satisfies

$$\frac{x}{5-x} = \frac{3}{4} .$$

If we solve this for x we find that find that $d(A,D) = x = \frac{15}{7}$, and hence we also have $d(B,D) = 5 - x = \frac{20}{7}$.■

4. The hypothesis on angle measures and $\angle DAC = \angle CAB$ imply that $\angle DAC \sim \angle CAB$ by the **AA** Similarity Theorem. This in turn implies that

$$\frac{d(A,B)}{d(A,C)} = \frac{d(A,C)}{d(A,D)}$$

and if we multiply by both denominators we find that $d(A,C)^2 = d(A,B) \cdot d(A,D)$, which is what we wanted to prove.■

5. To simplify the algebraic expressions let $a = d(A,X)$, $b = d(B,Y)$, $c = d(C,Z)$, $u = d(A,Z)$ and $v = d(Z,B)$. Since $A * Z * B$ holds, it follows that $d(A,B) = u + v$.

Since CZ is parallel to AX, we have $\Delta BCZ \sim \Delta BAX$, so that

$$\frac{c}{v} = \frac{b}{u+v}$$

and likewise since CZ is parallel to BY, we have $\Delta ACZ \sim \Delta ABY$, so that

$$\frac{c}{u} = \frac{a}{u+v} .$$

Since $p/q = r/s$ if and only if $p/r = a/x$, the preceding proportions imply

$$\frac{c}{a} = \frac{u}{u+v} , \qquad \frac{c}{b} = \frac{v}{u+v}$$

and if we add these equations we obtain

$$c \cdot \left(\frac{1}{a} + \frac{1}{b}\right) = \frac{u+v}{u+v} = 1$$

and if we divide both sides of this equation by c we obtain

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$$

which is equivalent to the equation in the conclusion of the exercise.■

Remark. This picture is closely related to the derivation of the *thin lens formula* in elementary physics. An interactive derivation of this formula is given at the following online site:

`http://www.hirophysics.com/Anime/thinlenseq.html`

The geometrical properties of similar triangles also play a significant role in other aspects of geometrical optics.

6. Let $k = a_1/b_1$. Then the proportionality assumption implies that $k = a_i/b_i$ for all i, and thus we also have $a_i = k\,b_i$ for all i. If we add all these equations together we obtain

$$a_1 + \cdots + a_n = k\,b_1 + \cdots + k\,b_n = k\,(b_1 + \cdots + b_n)$$

and if we solve this for k we obtain

$$k = \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n}$$

which is what we wanted to prove.■

7. (*i*) Let $k > 0$ be the ratio of similitude. Then we have $k \cdot d(B,C) \leq k \cdot d(A,C) \leq k \cdot d(A,B)$, and since we also have $d(E,F) = k \cdot d(B,C)$, $d(D,F) = k \cdot d(A,C)$, and $d(D,E) = k \cdot d(A,B)$, the inequalities in the exercise follow immediately.■

(*ii*) Suppose the vertices are labeled so that $d(B,C) \leq d(A,C) \leq d(A,B)$; we may always do this by taking suitable and consistent reorderings X, Y, Z and X', Y', Z'. of A, B, C and A', B', C'. Following the hint, we split the argument up into cases depending upon whether the triangles are equilateral, isosceles with the base shorter than the legs, isosceles with the legs shorter than the base, or scalene (no two sides have equal length).

Suppose first that the triangle is scalene. Then by the first part we know that $[BC]$ and $[B'C']$ are the shortest sides of the triangle, and since the triangle is scalene we know that these two sides must be the same, so that $\{B,C\} = \{B',C'\}$ and $d(B,C) = d(B',C')$. Since we have $k \cdot d(B,C) = d(B',C')$ by construction, it follows that k must be equal to 1.

Suppose next that we have $d(B,C) < d(A,C) = d(A,B)$, so that we also have $d(B',C') < d(A',C') = d(A',B')$. Once again we must have $\{B,C\} = \{B',C'\}$, and as in the preceding paragraph this implies $k = 1$.

Now suppose that we have $d(B,C) = d(A,C) < d(A,B)$, so that we also have $d(B',C') = d(A',C') < d(A',B')$. In this case we must have $\{A,B\} = \{A',B'\}$, and as in the preceding paragraphs this implies $k = 1$.

Finally, suppose that all three sides have equal length. Since all lengths in sight are equal we must have $d(A,B) = d(A',B')$, and from this we conclude that $k = 1$ also holds in this remaining case.■

8. Let k be the ratio of similitude, so that

$$\frac{d(D,E)}{d(A,B)} = \frac{d(E,F)}{d(B,C)} = \frac{d(D,F)}{d(A,C)} = k$$

and note that $\angle GBA = \angle CBA$ and $\angle HED = \angle FED$, so that $|\angle ABG| = |\angle DEH|$. Since G and H are midpoints of $[BC]$ and $[EF]$, we have $d(B,G) = \frac{1}{2}\,d(B,C)$ and $d(E,H) = \frac{1}{2}\,d(E,F)$, so that

$$\frac{d(E,H)}{d(B,G)} = \frac{d(E,F)}{d(B,C)} = k\ .$$

Combining these, we can apply the **SAS** similarity theorem to conclude that $\Delta ABG \sim \Delta DEH$ with ratio of similitude k.■

9. If $y = d(A, X)$, then the Angle Bisector Theorem implies that

$$\frac{y}{13 - y} \quad = \quad \frac{d(A, C)}{d(B, C)} \quad = \quad \frac{5}{12} \ .$$

Clearing this of fractions, we obtain the equation $12y = 65 - 5y$, so that $7y = 65$ and $y = 65/7 = 9\,\frac{2}{7}$.■

10. Let Q be the origin, so that $[AQ$ bisects $\angle BAC$ and the incenter J lies on the line AQ. In fact, we have $Q * J * A$ and of course $[AJ$ is also the bisector of $\angle QBA$ because the latter is just $\angle CBA$. Therefore, if $J = (0, y)$ then the Angle Bisector Theorem implies that

$$\frac{y}{h - y} \quad = \quad \frac{d(B, Q}{d(B, A)} \quad = \quad \frac{x}{\sqrt{x^2 + h^2}} \ .$$

If we clear this of fractions we obtain the equation

$$\left(\sqrt{x^2 + h^2}\right) y \quad = \quad (h - y)\, x$$

and if we solve this for y we find that

$$y \quad = \quad \frac{hx}{x + \sqrt{x^2 + h^2}} \ .$$

To check this formula, let us see what it yields if we have an equilateral triangle, so that $h = x\sqrt{3}$ or equivalently $x = h/\sqrt{3}$. In this case we see that $y = h/3$, which is the expected answer.■

III.6 : Circles and classical constructions

1. Let r be the radius of the circle. Then we have

$$d(Q, G)^2 \quad = \quad r^2 \ - \ \frac{d(A, B)^2}{4} \ , \qquad d(Q, H)^2 \quad = \quad r^2 \ - \ \frac{d(C, D)^2}{4}$$

and from these we see that $d(Q, G) = d(Q, H)$ if and only if $d(A, B) = d(C, D)$, while $d(Q, G) < d(Q, H)$ if and only if $d(A, B) > d(C, D)$.■

2. Suppose that L meets Γ at two points X and Y. Then ΔQXY is an isosceles triangle, so that $|\angle QXY| = |\angle QYX|$. Since a triangle has at most one angle that is not acute, it follows that the common measurement of the given two angles is less than 90° and hence $L = XY$ cannot be perpendicular to QX or QY. Therefore we know that QX must be perpendicular to L if X is the only point which lies on both the line L and the circle Γ.

Conversely, suppose that $QX \perp L$, and let W be any other point of L. Then Q, X and W form the vertices of a right triangle with the right angle at X, and since the hypotenuse of a right triangle is longer than either of the other sides, it follows that $d(Q, W) > d(Q, X)$, and hence the only point in $L \cap \Gamma$ is X.■

3. By the previous exercise we know that $QA \perp XA$ and $QB \perp XB$. Since $d(Q, A) = d(Q, B)$, we may apply the hypotenuse-side congruence theorem for right triangles to conclude that $\Delta QAX \cong \Delta QBX$, and this in turn implies that $d(A, X) = d(B, X)$.■

4. Following the hint, we first note that the line AX meets Γ in exactly two points because X lies in the interior of Γ. Denote these points by B and C. By Theorem III.3.8, it follows that $B * A * C$ is true. If $B \in (AX$, then B lies on $[AX \cap \Gamma$ but C does not, so $(AX \cap \Gamma = \{B\}$. On the other hand, if $B \notin (AX$, then we must have $C \in (AX$, so that C lies on $[AX \cap \Gamma$ but B does not, which means that $(AC \cap \Gamma = \{C\}$. Thus in either case we know that $(AX$ and Γ have exactly one point in common.■

5. We shall give the reasons for the steps in order.

(1) This is true by Proposition II.3.1.

(2) The hypotheses and (1) show that **SAS** applies.

(3) The first equation follows from (2), while the second follows from the hypotheses.

(4) This follows from (3) and the definition of the circle.

(5) This follows from the hypotheses and the definition of the interior of a circle.

(6) Apply Exercise 1.

(7) The statement $C \in (AG$ follows because $G \in (AC$, which implies that $(AC = (AG$, and the statement $C \in \Gamma$ follows because $d(B, C)$ is the radius of Γ.

(8) We know that $d(A, G) = d(A, C)$ and there is a **unique** point $Y \in (AG = AC$ at a given positive distance from A.

(9) The first statement follows because $C = G$, and the second follows by combining this with statement (2).■

6. Let Y be the unique point on $(QA$ which is also on the circle.

FIRST CASE. Suppose that A lies in the interior of the circle so that we have $Q * A * Y$. — Let X be any other point on the circle. If X lies on the line $QA = QY$, then we have $X * Q * Y$, and since $Q * A * Y$ also holds it follows that $X * Q * A$; clearly $d(A, X) > d(Q, Y) = d(Q, Y) > d(A, Y)$ in this case. Suppose now that X does not lie on the given line. Then by the Triangle Inequality we have $d(Q, A) + d(A, X) > d(Q, X) = d(Q, Y)$. But we also have $d(Q, Y) = d(Q, A) + d(A, Y)$. If we combine these inequalities we find that $d(Q, A) + d(A, X) > d(Q, A) + d(A, Y)$, which implies $d(A, X) > d(A, Y)$. Thus $d(Q, X) > d(Q, Y)$ for all points X on the circle other than Y.■

SECOND CASE. Suppose that A lies in the exterior of the circle so that we have $Q * Y * A$. — Let X be any other point on the circle. If X lies on the line $QA = QY$, then we have $X * Q * Y$, and since $Q * Y * A$ also holds it follows that $X * Y * A$; clearly $d(A, X) > d(A, Y)$ in this case. Suppose now that X does not lie on the given line. Then by the Triangle Inequality we have $d(Q, X) + d(X, A) > d(Q, A) = d(Q, Y) + d(Y, A)$. Since $d(Q, X) = d(Q, Y)$, this inequality string yields $d(X, A) > d(Y, A)$. Thus $d(Q, X) > d(Q, Y)$ for all points X on the circle other than Y.■

7. If Y and Z are the points of the two circles that are closest to X, then the preceding exercise implies that Y and Z both lie on the open ray $(QX$. The point is equidistant from the two circles if and only if X is the midpoint of $[YZ]$, so that we have $Q * Y * Z$ and $Y * X * Z$. It follows that $d(Q, X) = \frac{1}{2}(p + q)$. Since there is a unique such point Q on every ray with origin Q, it follows that the set of points in question is the circle with center Q and radius $\frac{1}{2}(p + q)$.■

8. Let L be a line, let A and F be distinct points of L, and let D be a point of the given plane P such that $L \subset P$ but $D \notin L$. By the Protractor Postulate for angular measurement there is a unique ray $[AE$ such that $(AE$ is contained in the side of L containing D and $|\angle EAF| = \alpha°$. By Proposition II.3.1 there are are points $B \in (AE$ and $C \in (AF$ such that $d(A, B) = c$ and

$d(A,C) = b$. These three points are not collinear because B does not lie on $L = AC$, and thus the points form the vertices of a triangle with the prescribed measurement data.■

9. Following the hint, suppose we have a line L and a circle with center Q and radius $r > 0$ such that L contains three points of the circle. CASE 1. *Suppose that* $Q \in L$. — In this case take a ruler function for L, and suppose that $Q \leftrightarrow q$ under this function. Then the intersection of L with the circle consists of exactly two points; namely, the points which correspond to the real numbers $q \pm r$. So it suffices to dispose of the remaining case:

CASE 2. *Suppose that* Q *does not lie on* L. — If L contains three points of L, then one of them is between the other two and therefore we can label them X, Y, Z such that $X * Y * Z$ is true. We then have isosceles triangles ΔQXY, ΔQYZ and ΔQXZ, and by three applications of the Isosceles Triangle Theorem we must have

$$|\angle QXZ = \angle QXY| \quad = \quad |\angle QZX = \angle QZY| \quad = \quad |\angle QYX| \quad = \quad |\angle QYZ| \; .$$

Since the sum of the measures of two angles of a triangle is less than $180°$, it follows that the common measurement of all the angles must be less than $90°$. In particular, this means that $|\angle QYX| + |\angle QYZ| < 90° + 90° = 180°$. On the other hand, by the Supplement Postulate for angle measures we also have $|\angle QYX| + |\angle QYZ| = 180°$, so we have a contradiction. The source of this contradiction is our assumption that the line and circle contain three distinct points in common, and therefore it follows that the latter cannot be true.■

10. The existence of a circle follows from the Circumcenter Theorem. To see uniqueness, it suffices to show that the center must be unique because if Q is the center of a circle containing a point A then the radius of the circle must be $d(Q,A)$. Now if Q is the center of a circle containing the three points A, B and C, then it follows that Q must lie on the perpendicular bisectors of AB, BC and AC. As in the proof of the Circumcenter Theorem, these bisectors are distinct lines (for example, if the perpendicular bisectors of AB and AC were the same, then this line would be perpendicular to AB and AC, and hence the latter two would be parallel, which they are certainly not because they meet at A). Since two distinct lines have at most one point in common, this shows that if P and Q are centers of circles containing the three given points, then $P = Q$.■

11. We can use Exercises III.4.3 and the definition of the circle Γ in this problem to show that if X belongs to Ω and $X \notin AB$, then $X \in \Gamma$. On the other hand, if $X = A$ or $X = B$, then $X \in \Gamma$ because $d(A,D) = d(B,D) = \frac{1}{2}d(A,B)$.

Conversely, suppose that $X \in \Gamma$. By Exercise 9 above we know that if $X \in L$ also holds then we must have $X = A$ or $X = B$. Suppose now that $X \in \Gamma$ but $X \notin AB$. We need to prove that $|\angle AXB| = 90°$. Our assumption implies that $d(A,D) = d(X,D) = d(B,D)$; furthermore, D lies on neither AX nor BX because $D \in AB$ is neither A nor B and

$$AB \cap AX \quad = \quad \{A\} \; , \qquad AB \cap BX \quad = \quad \{B\} \; .$$

Therefore the Isosceles Triangle Theorem implies that

$$|\angle DXA| \quad = \quad |\angle DAX| \; , \qquad |\angle DXB| \quad = \quad |\angle DBX| \; .$$

The Additivity Postulate for angle measure and the theorem on angle sums of triangles imply that

$$|\angle AXB| \quad = \quad |\angle AXD| \; + \; |\angle BXD| \; , \qquad |\angle AXB| \; + \; |\angle XAB| \; + \; |\angle XBA| \quad = \quad 180°$$

and noting that $\angle XAB = \angle XAD$ and $\angle XBA = \angle XBD$, we see that

$$2|\angle XAB| \ + \ 2|\angle XBA| \quad = \quad 180^\circ$$

and therefore we have

$$90^\circ \quad = \quad |\angle XAB| \ + \ |\angle XBA| \quad = \quad |\angle DXA| \ + \ |\angle DXB| \quad = \quad |\angle AXB|$$

which means that $AX \perp XB$, as required.■

12. If we write Z in the coordinate form (x, y), then $d(Z, L) = |x|$, $d(Z, M) = |y|$, and $d(Z, Q) = x^2 + y^2$, so the set we want is the set of all points whose coordinates satisfy $|x| + |y| = x^2 + y^2$. In situations like this, it is generally useful to split the problem into cases depending upon whether x and y are nonnegative or nonpositive; there are four possibilities. Since one obtains the same set if x and y are replaced by $\pm x$ and/or $\pm y$, it will be enough to consider points in the closed first quadrant which satisfy the conditions and then to take the corresponding sets in the other three quadrants (see the drawing in the `figures` file).

If we restrict attention to the first quadrant, then the defining equation becomes $x + y = x^2 + y^2$, which translates to

$$x^2 \ - \ x \ + \ y^2 \ - \ y \quad = \quad 0 \qquad \text{or} \qquad \left(x - \tfrac{1}{2}\right)^2 \ + \ \left((y - \tfrac{1}{2}\right)^2 \quad = \quad \tfrac{1}{4}$$

which is a circle with center $(\frac{1}{2}, \frac{1}{2}$ and radius $1/\sqrt{2}$. The line $x + y = 1$ is a diameter for this circle, and this means that the arc in the closed first quadrant is the semicircular arc which is totally contained in the closed first quadrant and with a diameter given the closed segment joining $(1, 0)$ to $(0, 1)$. By the symmetry conditions mentioned earlier, it follows that the set S is the union of the following semicircular arcs:

(1) The semicircular arc in the closed first quadrant with a diameter given the closed segment joining $(1, 0)$ to $(0, 1)$.

(2) The semicircular arc in the closed second quadrant with a diameter given the closed segment joining $(0, 1)$ to $(-1, 0)$.

(3) The semicircular arc in the closed third quadrant with a diameter given the closed segment joining $(-1, 0)$ to $(0, -1)$.

(4) The semicircular arc in the closed fourth quadrant with a diameter given the closed segment joining $(0, -1)$ to $(1, 0)$.

As indicated by the drawing in the `figures` file, the resulting set is a curve resembling the outline of a flower or four leaf clover.■

13. Following the hint, we shall first consider the special case where the points are situated as indicated. Then the circle is defined by the equation $x^2 + y^2 = a^2$ and its points are given by $(r \cos\theta,\, r \sin\theta)$ for some real number θ. We need to find the midpoints for the segments joining $A = (a, 0)$ to points $B(\theta)$ given by $(r \cos\theta,\, r \sin\theta)$ where $0 < \theta < 2\pi$. These are simply the points

$$M(\theta) \quad = \quad \left(\tfrac{1}{2}(a + a\cos\theta),\ \tfrac{1}{2}(a \sin\theta), \right) \ .$$

Note that this formula is even meaningful and valid if $\theta = 0$, for then it says that $B(0) = A$. We need to show that a point has the form $M(\theta)$ for some θ if and only if its coordinates satisfy the equation

$$\left(x - \frac{a}{2}\right)^2 \ + \ y^2 \quad = \quad \frac{a^2}{4} \ .$$

If P denotes the midpoint of the segment joining A to Q, then $P = \frac{1}{2}A$ and the left hand side is merely the square of the length of $M(\theta) - P$, and since $2(M(\theta) - P = 2M(\theta) - A = a(\cos\theta, \sin\theta)$ it follows that the left hand side is merely

$$\left|\tfrac{1}{2}\big(M(\theta) - A\right|^2 \quad = \quad \left|\, \tfrac{1}{2}a(\cos\theta, \sin\theta)\,\right|^2 \;\; \frac{a^2}{4} \; .$$

Therefore every point of Ω lies on the circle described in the exercise. Conversely, if a point X lies on that circle, then it has the form $X - P = \frac{1}{2}a(\cos\beta, \sin\beta)$ for some β, and one can now go backwards through the preceding argument to show that X is the midpoint of the segment joining A to $a(\cos\beta, \sin\beta)$. Therefore Ω is equal to the circle described in the exercise, provided we are in the special case.

To prove the general case, follow the hint. We have $A - Q = aU$ and $P - Q = \frac{1}{2}A$, and every point Z can be written in uniquely in the form $Z - Q = uU + vV$ for suitable scalars u, v. Then Γ is defined by the equation $u^2 + v^2 = a^2$, and we can use the argument above to prove that Ω is merely the circle of all points expressible as $X = P + \frac{a}{2}\cos\theta U + \frac{a}{2}\sin\theta V$ for some θ.■

Note. As indicated on page 406 of Greenberg, one can give a very short proof for this exercise using geometric transformations.

14. Let r be the radius of Γ so that

$$r \quad = \quad d(Q,A) \quad = \quad d(Q,B) \quad = \quad d(Q,C) \quad = \quad d(Q,D)$$

and also

$$d(Q,E) \quad = \quad \sqrt{r^2 - d(A,E)^2} \quad = \quad \sqrt{r^2 - \tfrac{1}{4}d(A,B)^2}\;, \qquad d(Q,F) \quad = \quad \sqrt{r^2 - d(C,F)^2} \quad =$$

$$\sqrt{r^2 - \tfrac{1}{4}d(C,D)^2}\;.$$

From these equations it follows that $d(Q,E) = d(Q,F)$ if and only if $d(A,B) = d(C,D)$.■

15. We shall follow the hint and use the setting of Exercises II.3.22.

In the earlier exercise we showed that $\Delta ABC \cong \Delta ADC$ by **SSS**. Therefore we have $|\angle BAC| = |\angle DAC|$ and $|\angle DCA| = |\angle BCA|$. Since we have a convex quadrilateral, the points A and C lie in the interiors of $\angle DAC$ and $\angle DAB$ respectively. If we combine the observations in the previous two sentences with the Additivity Postulate for angle measure, we obtain the following chains of equations:

$$|\angle DAB| \quad = \quad |\angle DAC| \;+\; |\angle BAC| \;\; 2|\angle DAC|\;, \qquad |\angle DCB| \quad = \quad |\angle DCA| \;+\; |\angle BCA| \;\; 2|\angle DCA|$$

Therefore the rays $[AC$ and $[CA$ bisect $\angle DAB$ and $\angle DCB$ respectively.

By the Crossbar Theorem, we know that the bisector of $\angle ABC$ meets (AC) in some point Q. Following the hint, we would like to show that $[DQ$ bisects $\angle ADC$. Suppose we can do this. Then by the characterization of angle bisectors, the point Q has the following properties:

(1) Since Q lies on the bisectors of $\angle DAB$ and $\angle DCB$, it follows that the distances from Q to AB and AD are equal (call the common value y) and also that the distances from Q to CB and CD are equal (call the common value z).

(2) Since Q lies on the bisector of $\angle ABC$, it follows that the distances from Q to AB and BC are equal (hence $y = z$).

If we combine these, we conclude that the distances from Q to AD and DC are also equal to $y = z$. But this implies that the circle with center Q and radius $y = z$ contains points on each of the four lines AB, BC, CD and DA such that the radii are perpendicular to these lines at the points of contact. In other words, the circle is tangent to all four lines. Note that the distance equalities also imply that Q lies on the angle bisector of $\angle ADC$.■

III.7 : Areas and volumes

1. Suppose that the vertices of the rhombus are given by $A,\ B,\ C,\ D$, and that the diagonals $[AC]$ and $[BD]$ meet at X. Let u be the area of the closed triangular region bounded by ΔABD, and let v be the area of the closed triangular region bounded by ΔCBD. Since we have a rhombus, it follows that $AC \perp BD$ and X is the midpoint of $[AC]$ and $[BD]$. Therefore it follows that

$$u \;=\; v \;=\; \left(\tfrac{1}{2}\, d(B,D) \cdot \tfrac{1}{2}\, d(A,C)\right) \;=\; \frac{1}{4} \cdot \Big(\, d(B,D) \cdot d(A,C) \,\Big)\ .$$

Now the area bounded by the rhombus is equal to $u + v$, and by the formula for $u = v$ this area is equal to $\frac{1}{2} \cdot d(B,D) \cdot d(A,C)$.■

2. In Heron's Formula we have $a = b = c$ and $s = \frac{3}{2}a$. Therefore Heron's Formula states that the area is equal to

$$\sqrt{\frac{3a}{2} \cdot \frac{a}{2} \cdot \frac{a}{2} \cdot \frac{a}{2}}$$

and this simplifies to

$$\frac{a^2\sqrt{3}}{4}\ .■$$

3. The answer is NO. Suppose that we are given a parallelogram $\diamondsuit ABCD$ such that $d(A,B) = d(C,D) = p$, $d(A,D) = d(B,C) = q$, and $|\angle DAB| = \theta$. Then the area of the region bounded by $\diamondsuit ABCD$ is equal to $p\,q\,\sin\theta$. Since we can construct a parallelogram of this sort for arbitrary positive integers p and q, and for arbitrary θ between 0 and 90 degrees, it is clear that one can have parallelograms that have sides with equal lengths but bound regions with different areas. — However, as noted after the proof of Heron's Formula, a theorem of Brahmagupta gives a formula for the area in terms of the lengths of the sides **provided** all of the vertices lie on a circle.■

4. Let W be the incenter of ΔABC, and let D, E and F be the feet of the perpendiculars from W to BC, AC and AB respectively. We then know that WD, WE and WF are altitudes for ΔWBC, ΔWAC, and ΔWAB respectively. The Star Property implies that the area μ of the solid region bounded by ΔABC is equal to the sum of the areas of the solid regions bounded by ΔWBC, ΔWAC, and ΔWAB; by the standard area formula, the latter three numbers are equal to $\frac{1}{2} \cdot d(B,C) \cdot r$, $\frac{1}{2} \cdot d(A,C) \cdot r$, and $\frac{1}{2} \cdot d(A,B) \cdot r$; therefore, if s is — as usual — equal to half of $d(A,B) + d(B,C) + d(A,C)$, it follows that the area μ is equal to $r \cdot s$. If we now combine Heron's Formula $\mu = \sqrt{s(s-a)(s-b)(s-c)}$ with $\mu = r\,s$ and solve for r, we obtain the formula stated in the exercise.■

5. If $D(r)$ denotes the area of a solid region bounded by a semicircular arc of radius r and the diameter joining its endpoints, then the area axioms imply that $2D(r) = \pi r^2$, and hence $D(r) = \frac{1}{2}\pi r^2$. By construction, the radii of the solid semicircular regions H_1 and H_2 with diameters

$[TU]$ and $[TS]$ are equal to 1, and the radius of the solid semicircular region K with diameter $[US]$ is equal to $\sqrt{2}$. Therefore the areas of H_1 and H_2 are equal to π, and the area of K is equal to $\pi\left(\sqrt{2}\right)^2 = 2\pi$. Using the presentations of these solid semicircular regions as unions of pieces which intersect in closed segments or circular arcs, we see that the areas of H_1 and H_2 are given by $B+C$, while the area of K is given by $2A+2B$. Combining these, we obtain the equations

$$\pi \quad = \quad \alpha(H_i) \quad = \quad B+C \ , \qquad 2\pi \quad = \quad \alpha(K) \quad = \quad 2A+2B \ .$$

If we divide both sides of the second equation by 2 and subtact the result from the first, we see that $0 = C - A$, so that $C = A$ as required.

Computing A is now fairly straightforward. The three points Q, T, S are noncollinear such that $SQ \perp TQ$ and $d(S,Q) = d(T,Q) = 1$, so the formula for the area of a solid region bounded by a triangle implies that

$$A \quad = \quad \frac{1}{2} \cdot d(S,Q) \cdot d(T,Q) \quad = \quad \frac{1}{2}$$

and therefore we also have that $C = \frac{1}{2}$.

Remarks. (1) One can also obtain the result in Exercise 5 using integral calculus, but this requires quite a bit of work.

(2) The blue shaded regions in the `figures` illustration are examples of *lunes* (the plural of *lune*, pronounced "loon" — the name arises because the special cases of *crescents*, for which both arcs lie in the same closed half-plane, look like phases of the moon); these are regions bounded by a pair of circular arcs. The work of Hippocrates also showed that certain other types of lunes had areas that could be evaluated by relatively simple expressions. Further results of this type were obtained in the late 18^{th} century by M. J. Wallenius (1731–1773) and later independently by others. Theorems of N. G. Chebotarev (1894–1947) and A. V. Dorodnov from the second quarter of the 20^{th} century showed that there were no other examples of lunes with areas that could be described in relatively simple terms (say by numbers that could be obtained from the positive integers by some classical geometric straightedge-and-compass constructions). References for further information are given in the following online file:

`http://math.ucr.edu/~res/math153/history02d.pdf`