SOLUTIONS TO EXERCISES FOR

MATHEMATICS 133 — Part 7

Winter 2009

NOTE ON ILLUSTRATIONS. Drawings for several of the solutions in this file are available in the following file:

`http://math.ucr.edu/~res/math133/math133solutions7figures.pdf`

V. Introduction to non – Euclidean geometry

V.1 : Facts from spherical geometry

1. The minor arc for A and B consists of A, B and all points of the circle Γ which are on the opposite side of as its center Q. Suppose that we have a pair of diametrically opposite points X and Y on the circle. We first claim that *both cannot lie on the union of* AB *with the side of* AB *opposite* Q.

First of all, both cannot lie on AB because AB meets Γ in exactly two points (A and B), and by hypotheses these points are not diametrically opposite. Suppose now that one of the points, say X is on the line AB. Then $X * Q * Y$ implies that Y and Q are on the same side of AB, so in particular it follows that Y cannot lie on the minor arc determined by A and B since this arc contains no points on the same side of AB as Q. Furthermore, if neither X nor Y lies on AB, suppose that both do line on the minor arc; it follows that they lie on the side of that line opposite Q. On the other hand, since $X * Q * Y$ holds it follows that Q also lies on this side, which yields a contradiction since Q cannot lie on the side opposite itself. This contradiction completes the proof that the minor arc does not contain a pair of diametrically opposite points.

On the other hand, the major arc contains many pairs of diametrically opposite points. For example, if we take C on the circle such that $A * Q * C$, then it follows that C and Q lie on the same side of AB and therefore C lies on the major arc determined by A and B.$\blacksquare$

2. As indicated in the hint, the center $\mathbf{w}$ of the circle is the foot of the perpendicular from the sphere center Q to the plane. The perpendicular direction to the plane $x + y + z = 1$ is given by the vector $(1, 1, 1)$, so $\mathbf{w}$ is the unique point $t(1, 1, 1)$ which lies on the plane $x + y + z = 1$. This means that $3t = 1$, so that $t = \frac{1}{3}$. The radius a of the circle is given by $\sqrt{r^2 - |\mathbf{w}|^2}$, where $r = 1$ is the radius of the sphere. Since $|\mathbf{w}|^2 = \frac{1}{9}$, it follows that $a = \sqrt{1 - \frac{1}{3}} = \sqrt{\frac{2}{3}}$.$\blacksquare$

3. Suppose that Σ_1 and Σ_2 are respectively defined by the equations

$$(1) \qquad |\mathbf{v} - \mathbf{a}|^2 = p^2$$

$$(2) \qquad |\mathbf{v} - \mathbf{b}|^2 = q^2$$

where $\mathbf{a} \neq \mathbf{b}$. If we subtract the second equation from the first, we obtain the following linear equation:

$$(3) \qquad 2 \cdot \langle \mathbf{x}, \mathbf{a} - \mathbf{b} \rangle \; + \; |\mathbf{a}|^2 \; - \; |\mathbf{b}|^2 \;\; = \;\; p^2 \; - \; q^2 \; .$$

This equation defines a plane whose normal direction is $\mathbf{a} - \mathbf{b}$, and it follows that if $\Sigma_1 \cap \Sigma_2$ is nonempty, then it is contained in a plane which is perpendicular to the line joining $\mathbf{a}$ and $\mathbf{b}$.

It follows immediately that the systems of equations {(1), (2)}, {(1), (3)}, and {(2), (3)} are equivalent. Therefore, if Π is the plane determined by (3), then we have

$$\Sigma_1 \cap \Sigma_2 \quad = \quad \Sigma_1 \cap \Pi \quad = \quad \Sigma_2 \cap \Pi \; .$$

This is what we wanted to prove.■

4. Let Q be the center of Σ.

Suppose first that $\Sigma \cap P = \{X\}$. Let Y be another point of P; we need to show that $QX \perp XY$. Let N be the plane determined by Q, X and Y; observe that these three points are not collinear because $Q \in XY$ implies $Q \in P$, which in turn implies that $\Sigma \cap P$ is a circle. Since $Q \in N$, it follows that $N \cap \Sigma$ is a circle that we shall call Γ_N. It follows immediately that $\Gamma_N \cap XY = \{X\}$ because this intersection is contained in $\Sigma \cap P = \{X\}$ and X lies on Γ_N and XY. By the theorems on circles, it follows that QX must be perpendicular to XY.

Conversely, suppose that $QX \perp P$. If Y is any other point in the plane P, then we are given that $Q \notin P$ and hence the points Q, X and Y must be noncollinear. In the right triangle ΔQXY we know that the length of the hypotenuse $d(Q, Y)$ is greater than $d(Q, X)$; but this means that Y cannot lie on the sphere Σ, whose center is Q and whose radius is $d(Q, X)$.■

V.2 : Attempts to prove Euclid's Fifth Postulate

1. (*a*) Follow the hint. First of all, there is a ruler function $f : L \to \mathbf{R}$ such that $f(A) < f(B)$. If g is an arbitrary ruler function, then either $g(A) < g(B)$ or else $g(A) > g(B)$. Take $g = f$ in the first case and $g = -f$ in the second.

Now choose Y to be the unique point on the line so that $f(Y) = f(A) + x$. It follows immediately that $d(Y, A) = x$, so we have to show that Y is on $(AB$ and in fact is the unique point such that $d(A, Y) = x$.

Suppose that Z is an arbitrary point such that $d(A, Z) = x$. Then $|f(A) - f(Z)| = x$ implies that $f(Z) = f(A) \pm x$, so that either $Z = Y$ or else Z is the unique point W such that $f(W) = f(A) - x$. It follows immediately that $d(W, B) = d(W, A) + d(A, B)$, so that $W * A * B$ is true. This means that $W \notin [AB$, and the only point on the latter ray with the prescribed distance from A would have to be Y.

There are now three cases, depending upon whether $x < d(A, B)$, $x = d(A, B)$, or $x > d(A, B)$. In the first case we have $f(A) < f(Y) < f(B)$, which implies that $d(A, B) = d(A, Y) + d(Y, B)$ and hence $A * Y * B$ is true. In the second we have $f(Y) = f(B)$ so that $Y = B$. Finaly, In the third case we have $f(A) < f(B) < f(Y)$, which implies that $d(A, Y) = d(A, B) + d(B, Y)$ and hence $A * B * Y$ is true. In each of the three cases we find that $Y \in [AB$, and the conclusions about ordering follow directly from the observations we have made in this paragraph.■

(*b*) Follow the hint and let $L = AB$ for suitable points $A \neq B$. By Exercise II.4.9 there is a point Y such that Y and X lie on opposite sides of AB with $d(A, X) = d(A, Y)$ and $|\angle XAB| = |\angle YAB|$. By **SAS** we have $\Delta XAB \cong \Delta YAB$, so that $d(B, X) = d(B, Y)$. Since A and B are equidistant from X and Y, it follows that $L = AB$ is the perpendicular bisector of $[XY]$. This proves the existence part of the theorem.

To prove uniqueness, suppose that there are two points C and D on L such that $XC \perp L$ and $XD \perp L$. Now let E be a point such that $C * D * E$. By the Exterior Angle Theorem we should have $|\angle XDE| > |\angle XCD|$; however, both angles in question are right angles, and thus we have a contradiction. The source of this contradiction is our assumption that there are two distinct perpendiculars from X to L, and thus it follows that there can only be one such perpendicular.■

Alternate approach to uniqueness. Here is a longer argument that is more self-contained because it does not require the Exterior Angle Theorem. — As before, suppose that we are given a line L and an external point X with two perpendiculars from X to L, and denote the feet of these perpendiculars by C and D. Then the Isosceles Triangle Theorem implies that $d(X, C) = d(X, D)$. Let Y and Z be points such that $X * C * Y$, $X * D * Z$, and $d(X, Y) = 2 \cdot d(X, C) = 2 \cdot d(X, D) = d(X, Z)$. By construction, both Y and Z lie on the side of L opposite X, and since $C \neq D$ it follows that $XY \neq XZ$. We now have $\Delta CDZ \cong \Delta CDX$ by **SAS**, and if we combine this with the isosceles triangle self-congruence $\Delta CDX \cong \Delta DCX$, we conclude that $ZC \perp CD = L$. On the other hand, since $X * C * Y$ we also have $YC = XC \perp CD = L$, so Y and Z are points on the same side of $L = CD$ such that $|\angle YCD| = |\angle ZCD| = 90°$. The Protractor Postulate for angle measures then implies that $[CZ = [CY$. This in turn implies that the lines $CZ = XZ$ and $CY = XY$ must be equal, contradicting our earlier conclusion that $XY \neq XZ$. The source of this contradiction is our assumption that there are two points C, D on L such that both XC and XD are perpendicular to L, and hence this must be false; in other words, there is at most one perpendicular to L through X.■

(c) Define D as suggested in the hint, so that $A * B * D$ and $d(A, D) = d(A, B) + d(B, C)$. It follows that $B \in \text{Int}\angle ACD$ and hence $|\angle ACD| > |\angle BCD|$. On the other hand, the Isosceles Triangle Theorem implies that $|\angle BCD| = |\angle BDC = \angle ADC|$, so that $|\angle ACD| > |\angle ADC|$. Since the longer side of a triangle is opposite the larger angle, it follows that $d(A, C) < d(A, D) = d(A, B) + d(B, C)$.■

(d) It suffices to consider the case where $d(A, C) > d(D, F)$, for if equality holds then one can derive the congruence conclusion by **SAS**, while if the reverse inequality holds one can proceed as in the originally stated case by reversing the roles of the two triangles.

Let $G \in (AC$ be such that $d(A, G) = d(D, F)$; it follows that $G \in (AC)$. Furthermore, by **SAS** we have $\Delta GAB \cong \Delta FDE$. This in turn implies that $|\angle GAB| = |\angle FDE| = |\angle CAB|$. However, since $A * G * C$ holds, we must have $|\angle GAB| > |\angle CAB|$, a contradiction. The source of this contradiction is our assumption that $d(A, C) > d(D, F)$, and hence the latter must be false. — As indicated in the first paragraph, one can similarly exclude the possibility that $d(A, C) < d(D, F)$, and thus the only possibility is $d(A, C) = d(D, F)$. As noted above, if we know this then we obtain the congruence relationship by **SAS**.■

V.3 : Neutral geometry

1. Let L be a line, and take points $A,\ B \in L$ such that $d(A, B) = q$. Let X be a point which does not lie on L, and consider the plane Π determined by L and X. By the Protractor Postulate there exist points U and V on the same side of L as X (in Π) such that $UA \perp AB$ and $VB \perp AB$. By Proposition II.3.1 there exist points $D \in (AU$ and $C \in (BV$ such that $d(A, D) = d(B, C) = p$. By Proposition V.3.5 it follows that $A,\ B,\ C,\ D$ determine the vertices of a convex quadrilateral, and by construction it is a Saccheri quadrilateral.■

2. Follow the hint. We have $k/h > 0$, so by density of the rationals there is a rational number m/n such that $m, n > 0$ and $0 < m/n < k/h$. But then we also have $0 < 1/n \leq m/n < k/h$, and these inequalities yield $h/n < k$. Since $n < 2^n$ for all positive integers n, it follows that $0 < h/2^n < h/n < k$, as stated in the exercise.■

3. By **SAS** we have $\Delta DAB \cong \Delta CBA$, so that the diagonals satisfy $d(B,D) = d(A,C)$. Therefore by **SSS** we have $\Delta CDA \cong \Delta DCB$, and this implies $|\angle CDA| = |\angle DCB|$.■

4. Let X and Y be the midpoints of $[AB]$ and $[CD]$ respectively. Then we have $\Delta DAX \cong \Delta CBX$ by **SAS**, so that $d(X,D) = d(X,C)$, so that XY is the perpendicular bisector of $[CD]$. Similarly, we have $\Delta ADY \cong \Delta BCY$ by **SAS** (this requires the previous exercise!), and therefore $d(Y,A) = d(Y,B)$, so that XY is the perpendicular bisector of $[AB]$. Therefore XY is perpendicular to AB and CD.■

4. We shall follow the hints. The proof of the inequality

$$d(G_1, G_n) \quad \leq \quad d(G_1, G_2) \ + \ \cdots \ + \ d(G_{n-1}, G_n)$$

proceeds by induction on $n \geq 2$. For $n = 2$ is simply says $d(G_1, G_2) \leq d(G_1, G_2)$. Suppose the statement is true for $n \geq 2$. Then by the Triangle Inequality we have

$$d(G_1, G_{n+1}) \ \leq d(G_1, G_n) + d(G_n, G_{n+1}) \ \leq \ \Big(d(G_1, G_2) + \cdots + d(G_{n-1}, G_n) \Big) + d(G_n, G_{n+1})$$

and hence the statement is also true for $n + 1$, completing the verification of the inductive step.

We continue by taking $X_0 = A$, $X_1 = B$, $Y_1 = C$ and $Y_0 = D$. Next, we take points X_k on $[X_0X_1$ such that $d(X_0, X_k) = k \cdot d(X_0, X_1) = k \cdot d(A, B)$. It then follows that we have $X_0 * X_k * X_{k+1}$ for all $k \geq 1$ and therefore we have $d(X_k, X_{k+1}) = d(X_0, X_1) = d(A, B)$. The idea is to construct a sequence of Saccheri quadrilaterals side by side so that the first one is the original quadrilateral and all have the same size and shape. To do this, for each $k \geq 2$ let Y_k be a point such that

(1) Y_k is on the same side of $X_0X_1 = AB$ as $Y_0 = D$ and $Y_1 = C$,

(2) Y_kX_k is perpendicular to $X_0X_1 = AB$, and

(3) $d(Y_k, X_k) = d(B, C) = d(A, D)$.

It follows that fore each k the points X_k, X_{k+1}, Y_k and Y_{k+1} are the vertices of a Saccheri quadrilateral with base given by $[X_kX_{k+1}]$.

The next step is to prove that the summits of these Saccheri quadrilaterals all have equal length, and to do this we must consider the auxiliary diagonal segments $[X_kY_{k-1}]$. It will suffice to prove that $d(Y_{k-1}, Y_k) = d(Y_k, Y_{k+1})$ for each k. First of all, we know that $\Delta Y_{k-1}Y_kX_k \cong \Delta Y_kX_kX_{k+1}$ by **SAS**. Therefore we have $d(X_k, Y_{k-1}) = d(X_{k+1}, Y_k)$ and also $|\angle X_{k-1}X_kY_{k-1}| = |\angle X_kX_{k+1}Y_k|$. Since Saccheri quadrilaterals are convex quadrilaterals, it follows that Y_{k-1} and Y_k lie in the interiors of of the respective right angles $\angle X_{k-1}X_kY_k$ and $\angle X_kX_{k+1}Y_{k+1}$. Therefore we have

$$|\angle Y_kX_kY_{k-1}| \quad = \quad 90 \ - \ |\angle X_{k-1}X_kY_{k-1}| \quad = \quad 90 \ - \ |\angle X_kX_{k+1}Y_k| \quad = \quad |\angle Y_{k+1}X_{k+1}Y_k| \ .$$

It follows that $|\Delta Y_kX_kY_{k-1}| \cong |\Delta Y_{k+1}X_{k+1}Y_k|$ by **SAS**, which in turn implies that $d(Y_{k-1}, Y_k) = d(Y_k, Y_{k+1})$ as claimed.

We now combine the conclusion of the preceding sentence with the inequality in the first paragraph for each positive integer n to obtain the inequalities

$$n \cdot d(A, B) = d(X_0, X_n) \quad \leq$$

$$d(X_0,Y_0) \ + \ d(Y_0,Y_1) \ + \ d(Y_1,Y_2) \ + \ \cdots \ + \ d(Y_n-1,Y_n) \ + \ d(Y_n,X_n) \ = \ n\cdot d(C,D) \ + \ 2\cdot d(A,D) \ .$$

Dividing both sides by n, we see that for each positive integer n we have

$$d(A,B) \ \leq \ d(C,D) \ + \ \frac{2}{n}\cdot d(A,D) \ .$$

We claim that $d(A,B) \leq d(C,D)$ follows from this; suppose to the contrary that $d(A,B) > d(C,D)$, and write $d(A,B) = d(C,D) + h$ where $h > 0$. We know that if n is sufficiently large then $2\cdot d(C,D)/n < h$, and for such values of n we conclude that

$$d(A,B) \ \leq \ d(C,D) \ + \ \frac{2}{n}\cdot d(A,B) \ < \ d(C,D) \ + \ h \ = \ d(A,B)$$

which is a contradiction. Therefore we must have $d(A,B) \leq d(C,D)$ as asserted in the exercise.■

6. The hypotheses imply that $d(A,B) = d(E,F)$ and $d(A,D) = d(B,C) = d(E,H) = d(F,G)$. By **SAS** we have $\Delta DAB \cong \Delta HEF$, and hence we also have $d(B,D) = d(F,H)$ and $|\angle DBA| = |\angle HFE|$. Since the $\diamondsuit ABCD$ and $\diamondsuit EFGH$ are Saccheri (hence convex) quadrilaterals, we know that $B \in \text{Int}\,\angle ABC$ and $H \in \text{Int}\,\angle EFG$. By additivity of angle measure, we then obtain

$$|\angle DBC| \ + \ 90 \ - \ |\angle DBA| \ = \ 90 \ - \ |\angle HFE| \ = \ |\angle HFG| \ .$$

Now we can use **SAS** to conclude that $\Delta DBC \cong \Delta HFG$, which implies that $d(C,D) = d(G,H)$ — in other words, the summits have equal length — and $|\angle DCB| = |\angle HGF$. Since the summit angles of a Saccheri quadrilateral have equal measures, it also follows that $|\angle ADC| = |\angle DCB| = |\angle HGF| = |\angle GHE|$, completing the proof.■

7. If we can prove the result with one of the two possible hypotheses on equal lengths, then the other will follow by interchanging the roles of the vertices, so we might as well assume that $d(A,B) = d(E,F)$.

By **SAS** we have $\Delta ABC \cong \Delta EFG$, and hence we also have $d(A,C) = d(E,G)$, $|\angle CAB| = |\angle GEF|$, and $|\angle ACB| = |\angle EGF|$. Since a Lambert quadrilateral is automatically a convex quadrilateral, it follows that $C \in \text{Int}\,\angle DAB$ and $G \in \text{Int}\,\angle HEF$; therefore by the additivity of angle measure we have

$$|\angle DAC| \ + \ 90 \ - \ |\angle CAB| \ = \ 90 \ - \ |\angle GEF| \ = \ |\angle GEH| \ .$$

Similarly, we have $A \in \text{Int}\,\angle BCD$ and $E \in \text{Int}\,\angle FGH$, so that

$$|\angle ACD| \ + \ 90 \ - \ |\angle ACB| \ = \ 90 \ - \ |\angle EGF| \ = \ |\angle EGH| \ .$$

Combining these, we see that $\Delta DAC \cong \Delta HEG$ by **ASA**, so that $d(C,D) = d(G,H)$, $d(A,D) = d(E,H)$ and $|\angle ADC| = |\angle EHG|$, completing the proof.■

8. By the result in the second exercise, it suffices to show that there is a right angle at D (because that will imply there is also a right angle at C). Since the summit and base have equal length, by **SSS** we must have $ADC \cong \Delta CBA$, so that $|\angle ADC| = |\angle CBA| = 90$.■

9. Following the hint, we begin by showing that it is enough to show that $d(A,D) \leq d(B,C)$. —- If we know this, then we can conclude that $d(A,B) \leq d(C,D)$ by reversing the roles of A and C in the discussion which follows.

We know there is a point $E \in (AB$ such that $d(A, E) = 2 \cdot d(A, B)$, and since $d(A, B) < d(A, E)$ it follows that $A*B*E$. Let $[EX$ be the unique ray such that $(EX$ lies on the same side of $AB = AE$ as D, and choose $F \in (EX$ so that $d(E, F) = d(A, D)$. Then the points A, E, F, D (in that order) form the vertices of a Saccheri quadrilateral with base $[AE]$.

Let G be the midpoint of $[DF]$. We claim that $G = C$. By Exercise 4 we know that BG is perpendicular to both AB and DF. Since BC is also perpendicular to AB it follows that $BC = BG$. Also, since both CD and GD are perpendicular $BC = BG$ and pass through D , it follows that $CD = GD$. Finally, since CD meets BC in C and GD meets BG in G, it follows that G and C must be the same point.

By the preceding paragraph we have $d(D, F) = 2 \cdot d(C, D)$. By Exercise 5 we have $d(A, E) \leq d(F, D)$, and if we combine these with the defining condition for E we have

$$2 \cdot d(A, B) \quad = \quad d(A, E) \quad \leq \quad d(D, F) \quad = \quad 2 \cdot d(C, D)$$

and if we divide these inequalities by 2 we obtain the desired relationship $d(A, B) \leq d(C, D)$.■

10. As in the preceding exercise, it is enough to prove that the quadrilateral is a rectangle if $d(A, B) = d(C, D)$.

It is fairly straightforward to give a proof of this statement which does not involve the construction of the preceding exercise by an argument similar to that for Exercise 8, but there is a very short proof using the Saccheri quadrilateral given above. — If we have $d(A, B) = d(C, D)$, then it follows that

$$d(A, E) \quad = \quad 2 \cdot d(A, B) \quad = \quad 2 \cdot d(C, D) \quad = \quad d(D, F)$$

and hence the auxiliary Saccheri quadrilateral is a rectangle. But this means that $\angle ADC = \angle ADF$ is a right angle, which in turn implies that the original Lambert quadrilateral is also a rectangle.■

11. By Exercise 1 we know that there is a Saccheri quadrilateral with vertices A, E, F, D (in that order) and base $[AE]$ such that $d(A, E) = 2q$ and $d(A, D) = p$. If B and C are the midpoints of $[AE]$ and $[DF]$ respectively, then we know that BC is perpendicular to both AE and DF, and hence the points A, B, C, D form the vertices of a Lambert quadrilateral with right angles at A, B, C. By construction we have $d(A, D) = p$ and $d(A, B) = \frac{1}{2} \cdot d(A, E) = q$.■

12. Following the hint, let $E \in (BA$ such that $d(B, E) = s$; since $d(B, E) < d(B, A)$ it follows that $B * E * A$, so that $E \in (AB)$. Let L be the perpendicular to AB containing E. Since L and AD are both perpendicular to AB, it follows that $L||AD$.

By Pasch's Theorem we know that L contains a point of either $[AD]$ or (BD). Since $L||AD$ it follows that L and (BD) have a point in common which we shall call G. Another application of Pasch's Theorem shows that L must also contain a point of either $[BC]$ or (CD). Since L and BC are both perpendicular to AB, the first of these is impossible, and therefore we must have some point $F \in L \cap (CD)$. Since L meets AB at E, it follows that E, B, C, F (in that order) form the vertices of a Lambert quadrilateral with right angles at E, B, C. By construction we have $d(E, B) = s$ and $d(B, C) = q$.■

13. Since E and F are midpoints of the sides containing them, we have $E \in (AC)$ and $F \in (AB)$, so that $\angle CAB = \angle EAF$; likewise, since E' and F' are midpoints of the sides containing them, we have $E' \in (A'C')$ and $F' \in (A'B')$, so that $\angle C'A'B' = \angle E'A'F'$. Therefore we have

$$|\angle EAF| \quad = \quad |\angle CAB| \quad = \quad |\angle C'A'B'| \quad = \quad |\angle E'A'F'| \; .$$

But we also have

$$\begin{aligned} d(A,E) &= \tfrac{1}{2}\, d(A,C) = \tfrac{1}{2}\, d(A',C') = d(A',E') \\ d(A,F) &= \tfrac{1}{2}\, d(A,B) = \tfrac{1}{2}\, d(A',B') = d(A',F') \end{aligned}$$

and therefore by **SAS** we have $\Delta AEF \cong \Delta A'E'F'$.

To prove the remaining two statements, first change letters so that A, B, C, D, E, F and A', B', C', D', E', F' become X, Y, Z, U, V, W and X', Y', Z', U', V', W' respectively. Then the hypothesis becomes $\Delta XYZ \cong \Delta X'Y'Z'$ and the conclusion becomes $\Delta XVW \cong \Delta X'V'W'$. To prove that $\Delta BFD \cong \Delta B'F'D'$, observe that if we take X, Y, Z and X', Y', Z' to be B, C, A and B', C', A' respectively, then U, V, W and U', V', W' become E, F, D and E', F', D' respectively. Since $\Delta ABC \cong \Delta A'B'C'$ implies $\Delta BCA \cong \Delta B'C'A'$, it follows that we also have $\Delta BFD \cong \Delta B'F'D'$.

Similarly, to prove that $\Delta CDE \cong \Delta C'D'E'$, observe that if we take X, Y, Z and X', Y', Z' to be C, A, B and C', A', B' respectively, then U, V, W and U', V', W' become F, D, E and F', D', E' respectively. Since $\Delta ABC \cong \Delta A'B'C'$ implies $\Delta CAB \cong \Delta C'A'B'$, it follows that we also have $\Delta CDE \cong \Delta C'D'E'$.

The preceding three arguments show that

$$d(E,F) = d(E',F')\ , \qquad d(F,D) = d(F',D')\ , \qquad d(D,E) = d(D',E')$$

and therefore by **SSS** we must also have $\Delta DEF \cong \Delta D'E'F'$.∎

14. If we are working in a Euclidean plane, then we also know that the distance between the midpoints of two sides is half the length of the third side, so that the following hold:

$$\begin{aligned} d(E,F) &= \tfrac{1}{2}d(B,C) = \tfrac{1}{2}d(B',C') = d(E',F') \\ d(F,D) &= \tfrac{1}{2}d(C,A) = \tfrac{1}{2}d(C',A') = d(F',D') \\ d(D,E) &= \tfrac{1}{2}d(A,B) = \tfrac{1}{2}d(A',B') = d(D',E') \end{aligned}$$

If we combine these with the midpoint conditions which are given in the problem, then by repeated applications of **SSS** we can conclude the following:

$$\begin{aligned} \Delta AFE &\cong \Delta DEF \cong \Delta D'E'F' \cong \Delta A'F'E' \\ \Delta BFD &\cong \Delta EDF \cong \Delta E'D'F' \cong \Delta B'F'D' \\ \Delta CDE &\cong \Delta FED \cong \Delta F'E'D' \cong \Delta C'D'E' \end{aligned}$$

Finally, we may reorder vertices to rewrite the preceding in the following form:

$$\Delta AFE \cong \Delta FBD \cong \Delta EDC \cong \Delta DEF =$$

$$\Delta D'E'F' \cong \Delta E'D'C' \cong \Delta F'B'D' \cong \Delta AF'E'$$

This explicitly displays the eight triangle congruences.∎

V.4 : Angle defects and related phenomena

1. Suppose first that we have a Saccheri quadrilateral $\diamondsuit ABCD$ in a hyperbolic plane with base $[AB]$. By Exercise V.3.5 above, we know that $d(A,B) \leq d(C,D)$, and furthermore by Exercise V.3.8 we know that if the Saccheri quadrilateral is a rectangle if equality holds. Since rectangles do not exist in a hyperbolic plane, we must have the strict inequality $d(A,B) < d(C,D)$.■

Now suppose that that we have a Lambert quadrilateral $\diamondsuit ABCD$ in a hyperbolic plane with right angles at A, B, C. By Exercise V.3.9 and V.3.10 we know that $d(A,B) \leq d(C,D)$ and $d(A,D) \leq d(B,C)$, and if either $d(A,B) = d(C,D)$ or $d(A,D) = d(B,C)$ then the Lambert quadrilateral is a rectangle. As above, since rectangles do not exist in a hyperbolic plane, we must have the strict inequalities $d(A,B) < d(C,D)$ and $d(A,D) < d(B,C)$.■

2. This follows fairly directly from Theorem V.4.9 in the notes. By Exercise 3 from the preceding section, we know that the lines containing the summit and base of the Saccheri quadrilateral have a common perpendicular, and the theorem from the notes says that the shortest distance from a point on one line to the other is realized at the points where the two parallel lines meet this common perpendicular. Since the lines containing the lateral sides of a Saccheri quadrilateral are perpendicular to the line containing the base, it follows that the length of a lateral side must be greater than the length of the segment joining the midpoints of the summit and base, for the line joining these two points is the common perpendicular.■

3. If we split a triangle ΔABC into two triangles by a segment $[BD]$ where $D \in (AC)$, then we have

$$\delta(\Delta ABC) \quad = \quad \delta(\Delta ABD) \; + \; \delta(\Delta ADC)$$

and since all numbers in sight are positive it follows that at least one of the numbers on the right hand side is less than or equal to $\frac{1}{2}\delta(\Delta ABC)$. By induction, for each n we can construct a triangle $\Delta X_nY_nZ_n$ such that $\delta(\Delta X_nY_nZ_n) \leq \delta(\Delta ABC)/2^n$. One can now use the first exercise from the preceding section to show there is some n for which the right hand side is less than ε.■

4. As in the proof of the Hyperbolic **AAA** Congruence Theorem we know that the defects satisfy $\delta(\Delta ADE) < \delta(\Delta ABC)$. If we apply the Isosceles Triangle Theorem and the definition of defect to both triangles we find that

$$180 \;-\; |\angle BAC| \;-\; 2|\angle ADE| \quad = \quad \delta(\Delta ADE) \quad < \quad \delta(\Delta ABC) \quad =$$

$$180 \;-\; |\angle BAC| \;-\; 2|\angle ABC|$$

and from this point one cqan use standard manipulations with inequalities to prove that $|\angle ADE| > |\angle ABC|$.■

5. Since equilateral triangles are equiangular, we know that $|\angle BAC| = |\angle ABC| = |\angle BCA|$; let us denote this common value by ξ. Since D, E and F are midpoints of the sides of an equilateral triangle, we know that

$$d(A,F) \quad = \quad d(F,B) \quad = \quad d(B,D) \quad = \quad d(D,C) \quad = \quad d(C,E) \quad = \quad d(E,A)$$

and therefore we have $\Delta AEF \cong \Delta BFD \cong \Delta CDE$ by **SAS**. All three of these smaller triangles are isosceles, so that we also have

$$|\angle AEF| \quad = \quad |\angle AFE| \quad = \quad |\angle BFD| \quad = \quad |\angle BDF| \quad = \quad |\angle CDE| \quad = \quad |\angle CED|$$

and we shall denote the common value by η.

The triangle congruences also imply

$$d(E,F) \quad = \quad d(F,D) \quad = \quad d(D,E)$$

and hence ΔDEF is also an equilateral triangle. Thus it is also equiangular, so let φ be the measure of the three vertex angles. The second relationship to proved in the exercise then translates to showing that $\varphi > \xi$.

Since we are working in hyperbolic geometry we know that the angle sum of, say, ΔAEF is less than 180 degrees, and if we substitute the values ξ and η into this inequality we find that $\xi + 2\eta < 180$.

A picture suggests that we should also have $\varphi + 2\eta = 180$, but we need to prove this. A key step in doing this is to show that E lies in the interior of $\angle DFA$. To prove this, first observe that the betweenness relations $C * E * A$ and $C * D * B$ imply that C, D and E all lie on the same side of AB. Next, the betweenness relations $A * F * B$ and $C * D * B$ imply that B lies on the side of FD opposite both C and A, so that A and C lie on the same side of DF. Finally, $E \in (AC)$ now implies that A and E must lie on the same side of DF, completing the requirements for E to lie in the interior of $\angle DFA$.

The preceding paragraph implies that $|\angle DFA| = |\angle DFE| + |\angle EFA| = \varphi + \eta$. Since $A * F * B$ holds, we also have

$$180 \quad = \quad |\angle DFA| \ + \ |\angle DFB| \quad = \quad \varphi \ + \ \eta \ + \ \eta \quad = \quad \varphi \ + \ 2 \cdot \eta$$

which was the claim at the beginning of the preceding paragraph. It now follows that

$$\xi \ + \ 2 \cdot \eta \quad < \quad 180 \quad = \quad \varphi \ + \ 2 \cdot \eta$$

which implies $\xi < \eta$, proving the inequality stated in the second assertion of the exercise.

Finally, we need to show that the isosceles triangle ΔAEF is not an equilateral triangle. However, the preceding exercise implies that

$$|\angle EFA| \quad > \quad |\angle ABC|$$

and since the right hand side is equal to $|\angle CAB = \angle EAF$, we can use Theorem III.2.5 (the larger angle is opposite the longer side) to conclude that $d(A,E) < d(F,A)$.■

6. We know that there is a ray $[DX$ such that $(DX$ lies on the same side of AB as C and $|\angle EDA| = |\angle CBA|$. The rays $[DX$ and $[BC$ cannot have a point in common, for if they met at some point Y then the Exterior Angle Theorem would imply $|\angle EDA| > |\angle CBA|$ and by construction these two numbers are equal.

By Pasch's Theorem the line DX must have a point in common with either $[BC]$ or (AC). Since $[DX$ and $[BC$ have no points in common by the preceding paragraph, it follows that there must be a point $E \in (AC) \cap DX$. Since $A * E * C$ is true, it follows that E and C lie on the same side of AB, so that $[DE = [DX$.

Since $E \in (AC)$ and $D \in (AB)$, the angle defects of ΔABC and ΔADE satisfy

$$\delta(\Delta ABC) \quad = \quad \delta(\Delta ADE) \ + \ \delta(\Delta EDC) \ + \ \delta(\Delta DBC)$$

so that $\delta(\Delta ADE) < \delta(\Delta ABC)$. On the other hand, by construction we have

$$\delta(\Delta ABC) \;-\; \delta(\Delta ADE) \;\;=\;\; |\angle AED| \;-\; |\angle ACB|$$

and since the left hand side is positive it follows that $|\angle AED| > |\angle ACB|$, which is what we wanted to prove.■

7. Suppose that the ray $[AC$ bisects $\angle DAB$. Then we have $|\angle CAD| = |\angle DAB| = 45°$. On the other hand, since ΔABC is an isosceles triangle with a right angle at B, it will follow that $|\angle ACB| = 45°$. In particular, this means that the angle defect of ΔABC is zero. This cannot happen in a hyperbolic plane, and therefore the ray $[AC$ cannot bisect $\angle DAB$.■

8. Follow the hint, so that B is a point not on a line L such that there are at least two parallel lines to L through B. One of the lines can be constructed by dropping a perpendicular from B to L whose foot we shall call Y, and then taking a line M which is perpendicular to BY and passes through B. Let N be a second line through B which is parallel to L.

Since L and M are parallel, all points of L lie on the same side of M. Since N contains points on both sides of M, it follows that there is some point A which lie on N and also on the same side of M as L. Note that $A \notin BY$, because $N \cap BY = \{B\}$ and $B \in M$. Since M contains points on both sides of BY, there is also a point $C \in M$ which lies on the side of BY which does not contain A (hence A and C lie on opposite sides of BY).

We claim that L is contained in the interior of $\angle ABC$. The first step is to show that Y lies in the interior of this angle. By construction we know that $Y \in L$ and since L and a lie on the same side of M, it follows that Y and A lie on the same side of $M = BC$. On the other hand, since A and C lie on opposite sides of BY we know there is a point $Z \in (AC) \cap BY$. It follows that A and Z lie on the same side of $BC = M$, and since A and Y also lie on the same side of M it follows that $(BY = (BZ$. But this means that C, Z and Y must all lie on the same side of $N = AB$. Thus we have shown that Y lies in the interior of $\angle ABC$.

Since L does not have any points in common with either M or N, it follows that all points of L lie on the same side of each line. We have seen that $Y \in L$ lies on the same side of $M = BC$ as A and on the same side of $N = AB$ as C, and therefore the same must be true for every point of L. But this means that L is contained in the interior of $\angle ABC$.■