SOLUTIONS TO EXERCISES FOR

MATHEMATICS 133 — Part 8

Winter 2009

V.7 : Non–Euclidean geometry in modern mathematics

1. We first consider the special case where $\mathbf{a} = \mathbf{0}$. In this case we have

$$T(\mathbf{v}) \quad = \quad \frac{r^2}{|\mathbf{v}|^2} \cdot \mathbf{v}$$

so that $T(\mathbf{v})$ is a positive multiple of $\mathbf{v}$ and $|T(\mathbf{v})| \cdot |\mathbf{v}| = r^2$.

We shall now use this to prove that $T\big(T(\mathbf{v})\big) = \mathbf{v}$ for all $\mathbf{v}$: If $\mathbf{w} = T(\mathbf{v})$, then it follows that $T(\mathbf{w})$ is a positive multiple of $\mathbf{w}$, and since the latter is a positive multiple of $\mathbf{v}$, it follows that $T\big(T(\mathbf{v})\big) = \mathbf{v} = T(\mathbf{w})$ is a positive multiple of $\mathbf{v}$. Also, we have

$$|T(\mathbf{w})| \quad = \quad \frac{r^2}{|\mathbf{w}|} \quad = \quad \frac{r^2}{r^2/|\mathbf{v}|} \quad = \quad |\mathbf{v}|$$

and if we combine this with the observation in the preceding sentence we see that $T\big(T(\mathbf{v})\big) = \mathbf{v}$.

We can use this to prove T is 1–1 and onto as follows. If $T(\mathbf{x}) = T(\mathbf{y})$, then if we apply T to both sides we see that $\mathbf{x} = \mathbf{y}$. Similarly, if we are given any vector $\mathbf{w}$, then the identity $T\big(T(\mathbf{w})\big) = \mathbf{w}$ shows that $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ (namely, $T(\mathbf{w})$). The identity at the end of the previous paragraph also shows that if $\mathbf{w} = T(\mathbf{v})$, then $T(\mathbf{w}) = \mathbf{v}$, and conversely if $\mathbf{v} = T(\mathbf{w})$, then $T(\mathbf{v}) = \mathbf{w}$. These conditions show that T is an inverse function to itself.

We now turn to the general case. If T_0 is the map $T_0(\mathbf{x}) = r^2|\mathbf{x}|^{-2}\,\mathbf{x}$ considered in the previous paragraph, then we have

$$T(\mathbf{v}) \quad = \quad \mathbf{a} \; + \; T_0(\mathbf{v} - \mathbf{a})$$

and if we apply T twice we then obtain

$$T\big(T(\mathbf{v})\big) \quad = \quad T\big(\mathbf{a} + T_0(\mathbf{v} - \mathbf{a})\big) \quad =$$

$$\mathbf{a} \; + \; T_0\big(\mathbf{a} + T_0(\mathbf{v} - \mathbf{a}) - \mathbf{a}\big) \quad = \quad \mathbf{a} \; + \; T_0\big(T_0(\mathbf{v} - \mathbf{a})\big) \quad = \quad \mathbf{a} \; + \; (\mathbf{v} - \mathbf{a}) \quad = \quad \mathbf{v}\ .$$

Since applying T twice again yields the identity, it follows as before that T must be its own inverse.

2. By the definitions we have

$$T(\mathbf{v} \quad = \quad \frac{1}{|\mathbf{v}|^2 \cdot \mathbf{v}}\ , \qquad S(\mathbf{v}) \quad = \quad \frac{r^2}{|\mathbf{v}|^2 \cdot \mathbf{v}}$$

and therefore straightforward computation yields

$$r\,T(r^{-1}\mathbf{v}) \quad = \quad r \cdot \left(\frac{1}{|r^{-1}\mathbf{v}|^2} \cdot r^{-1}\,\mathbf{v}\right) \quad = \quad r\,\left(\frac{1}{r^{-1}|\mathbf{v}|^2}\right)\mathbf{v} \quad = \quad \frac{r^2}{|\mathbf{v}|^2}\,\mathbf{v} \quad = \quad S(\mathbf{v})$$

which is the identity we wanted to prove.■

3. Following the hint, we split the argument into two cases.

CASE 1. *The center of* Γ_1 *is the origin, and the radius is* k. — Then $\mathbf{v} \in \Gamma_1$ implies that $|\mathbf{v}| = k$ and hence $|T(\mathbf{v})| = 1/k$, so that T maps $\Gamma_1 = \Gamma(k, \mathbf{0})$ to $\Gamma'_2 = \Gamma(1/k, \mathbf{0})$. It follows that $\Gamma_2 \subset \Gamma'_2$. Since

$$k \quad = \quad \frac{1}{\left(\frac{1}{k}\right)}$$

it follows that T also maps $\Gamma'_2 = \Gamma(1/k, \mathbf{0})$ to $\Gamma_1 = \Gamma(k, \mathbf{0})$. To see that the image of Γ_1 under T is all of Γ'_2, note that the identity $T\big(T(\mathbf{w})\big) = \mathbf{w}$ shows that every $\mathbf{w} \in \Gamma'_2$ is the image under T of the point $T(\mathbf{w}) \in \Gamma_1$.■

CASE 2. *The center of* Γ_1 *is some point* $\mathbf{a} \neq \mathbf{0}$. — Once again let k be the radius of the circle. Since $\mathbf{0}$ does not lie on Γ_1 it follows that the radius k is not equal to $k = |\mathbf{0} - \mathbf{a}| = |\mathbf{a}|$; this means that $q = |\mathbf{a}|^2 - k^2$ is nonzero. If $\mathbf{v} \in \Gamma_1$ then we have

$$k^2 \quad = \quad |\mathbf{v} - \mathbf{a}|^2 \quad = \quad |\mathbf{v}|^2 - 2\mathbf{v} \cdot \mathbf{a} + |\mathbf{a}|^2$$

so that $2\mathbf{v} \cdot \mathbf{a} = |\mathbf{v}|^2 + q$. It now follows that

$$\left| T(\mathbf{v}) \; - \; q^{-1}\mathbf{a} \right|^2 \quad = \quad \left| \frac{1}{|\mathbf{v}|^2|} \, \mathbf{v} \; - \; q^{-1}\mathbf{a} \right|^2 \quad =$$

$$\frac{1}{|\mathbf{v}|^2} \; + \; \frac{|\mathbf{a}|^2}{q^2} \; - \; \frac{2\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|^2 \, q^2} \; .$$

If we now put everything over a common denominator and use our previously derived formula for $2\mathbf{v} \cdot \mathbf{a}$, we find that the right hand side is equal to

$$\frac{q^2 + |\mathbf{v}|^2|\mathbf{a}|^2 - |\mathbf{v}|^2 q^2 - q^2}{|\mathbf{v}|^2 q^2}$$

and if we simplify this we find that the quantity in question is equal to k^2/q^2. This means that the mapping T sends Γ_1 to the circle Γ'_2 whose center is $\mathbf{b} = q^{-1}\mathbf{a}$ and whose radius is $m = |q|^{-1}k$.

The vector $\mathbf{0}$ also does not lie on Γ'_2 because

$$m \quad = \quad |q|^{-1}k \quad \neq \quad |q|^{-1}|\mathbf{a}| \quad = \quad |q^{-1}\mathbf{a}| \quad = \quad |\mathbf{b}|$$

and as in the preceding discussion $\mathbf{0}$ lies on a circle centered at $\mathbf{b}$ only if the circle's radius is $|\mathbf{b}|$. Therefore we can apply the reasoning of the preceding paragraph to conclude that T maps Γ'_2 into the circle whose center is $p^{-1}\mathbf{b} = p^{-1}q^{-1}\mathbf{a}$ and whose radius is $|p|^{-1}m = |p|^{-1}|q|^{-1}k$. If we can show that $p = 1/q$, then it will follow that T maps Γ'_2 into the original circle Γ_1, and as in Case 1 we can use $T\big(T(\mathbf{w})\big) = \mathbf{w}$ to conclude that the image of Γ_1 under T is all of Γ'_2.

But now we have

$$p \;=\; |\mathbf{b}|^2 \;-\; m^2 \quad = \quad \frac{|\mathbf{a}|^2}{q^2} \;-\; \frac{k^2}{q^2} \quad = \quad \frac{|\mathbf{a}|^2 - k^2}{q^2} \quad = \quad \frac{q}{q^2}$$

and since the right hand side simplifies to $1/q$, we have completed the remaining step in the argument.■

4. We first show that T maps the given circle to the given line. By definition, the circle Γ_1 is defined by the equation $x^2 + (y-b)^2 = b^2$ and if we expand the left hand side we obtain the equivalent formulation $x^2+y^2-2by+b^2 = b^2$; subtacting $b^2 - 2by$ from both sides shows that the circle is also defined by the equation $x^2+y^2 = 2by$. Therefore, if $(u,v) = T(x,y)$, then by definition we have

$$(u,v) \quad = \quad \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$$

and if $(x,y) \in \Gamma_1$ then we may apply the equation at the end of the previous sentence to rewrite the right hand side as

$$\left(\frac{x}{2by}, \frac{y}{2by} \right) \quad = \quad \left(\frac{x}{2by}, \frac{1}{2b} \right) .$$

Therefore the image of Γ_1 lies on the horizontal line L defined by $y = 1/(2b)$.

Suppose now that we are given $(u,v) \in L$, so that $v = 1/(2b)$. We need to show that $(x,y) = T(u,v)$ lies on Γ_1, which translates to the algebraic equation $x^2+y^2 = 2by$. Since

$$(x,y) \quad = \quad T(u,v) \quad = \quad \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2} \right)$$

and the definition of T implies that

$$x^2+y^2 \quad = \quad \frac{1}{u^2+v^2}$$

if $v = 1/(2b)$ we obtain the following:

$$x^2+y^2 \quad = \quad \frac{1}{u^2+(1/2b)^2} \quad = \quad \frac{4b^2}{4b^2u^2+1} \ , \qquad y \quad = \quad \frac{v}{u^2+v^2} \quad = \quad \frac{1}{2b} \cdot \frac{4b^2}{4b^2u^2+1}$$

Comparing these results, we see that $x^2+y^2 = 2by$ and therefore $T(u,v) = (x,y) \in \Gamma_1$, so that T sends points of Γ_1 to points of L and vice versa. As in the proofs of preceding exercises, it follows that the images of Γ_1 and L are L and Γ_1 respectively.$\blacksquare$

5. The two circles meet at the two points $(1,0)$ and $(0,1)$. This seems obvious from the picture, and it follows algebraically by solving the simultaneous set of quadratic equations $x^2+y^2 = 1$ and $(x-1)^2+(y-1)^2 = 1$. The tangent lines to the first circle at the intersection points are $(1,0) + V$ and $(0,1) + H$, where V and H are the vertical and horizontal 1–dimensional vector subspaces spanned by $(0,1)$ and $(1,0)$ respectively. Similarly, the tangent lines to the second circle at the intersection points are $(1,0) + H$ and $(0,1) + V$. These follow because the tangent is perpendicular to the radius at the point of contact.

Next, let C and D be the two points where Γ_1 meets the line $y = x$. The first coordinates of these points satisfy the equation $2\,(x-1)^2 = 1$, so that $x - 1 = \pm\frac{1}{2}\sqrt{2}$ and hence $x = 1 \pm \frac{1}{2}\sqrt{2}$. The second coordinates are then given by $y = x$.

Let u and v be the two values for $x = y$. We claim that the lengths of the two vectors (u,u) and (v,v) are reciprocals of each other. Since the length of an arbitrary vector of the form (w,w) is $|w|\,\sqrt{2}$, it follows that the lengths of (u,u) and (v,v) are equal to

$$\left(1 \pm \frac{1}{2}\sqrt{2}\right) \cdot \sqrt{2} \quad = \quad \sqrt{2} \pm 1$$

so that the product of these two lengths is

$$\left(\sqrt{2}\ +\ 1\right)\cdot\left(\sqrt{2}\ -\ 1\right) \quad = \quad 1$$

which is what we wanted to prove.

By the computations in the preceding discussion, we know that the points C and D are mapped to each other under the inversion map T. Since Γ_1 does not contain the origin (because $(0-1)^2+(0-1)^2 = 2 \neq 1$), by Exercises 3 and 4 it follows that T maps Γ_1 into a circle which contains the points $A = T(A), B = T(B), C = T(D), D = T(C)$. Now there is a unique circle containing the noncollinear points A, B, C, and since both Γ_1 and its image contain all three of them it follows that Γ_1 and its image must be the same circle.∎

6. (*i*) Start off by following the hint. The center $(c, 0)$ of the circle Γ_1 must be the midpoint of the line segment joining $(a, 0)$ to $(b, 0)$, and by the formula for finding midpoints this is $\frac{1}{2}(a+b)$. On the other hand, the diameter $2r$ of this circle is merely $b-a$, and hence $r = \frac{1}{2}(b-a)$. As noted in the statement at the end of the exercise, there are right angles at B and C if and only if $r^2+1 = c^2$.

Since we have expressed r and c in terms of a and b, we can also write c^2 and r^2 in terms of these variables:

$$c^2 \quad = \quad \frac{b^2+2ab+a^2}{4}\ , \qquad r^2 \quad = \quad \frac{b^2-2ab+a^2}{4}$$

Therefore we obtain the following formula for c^2-r^2-1:

$$c^2\ -\ r^2\ -\ 1 \quad = \quad \frac{b^2+2ab+a^2}{4}\ -\ \frac{b^2-2ab+a^2}{4}\ -\ 1 \quad =$$

$$\frac{4ab}{4}\ -\ 1 \quad = \quad ab\ -\ 1$$

It follows that $r^2+1 = c^2$, which is equivalent to $c^2-r^2-1 = 0$, is satisfied if and only if $ab = 1$, or in other words if and only if a and b are reciprocals of each other.∎

(*ii*) If $Y = (y, 0)$ where $y > 0$, then by the first part of the exercise the circle with center $(c, 0)$ and radius r, where

$$c \quad = \quad \frac{1}{2}\left(y\ +\ \frac{1}{y}\right)\ , \qquad r \quad = \quad \frac{1}{2}\left(\frac{1}{y}\ -\ y\right)$$

will satisfy the specified conditions. More generally, if Y is an arbitrary point, then we may write it as $y(\cos\theta, \sin\theta)$ for some θ and some y satisfying $0 < y < 1$, and if we define r and c as before, then the circle with center $c(\cos\theta, \sin\theta)$ and radius r will have the desired properties.∎

(*iii*) This is similar to the last part of the proof of the previous exercise. Let Y be as before, and let $Z = T(Y)$. Then T maps B and C to themselves and switches Y and Z. Now the circle Γ_1 does not contain the origin because $r \neq c$ (the difference $c-r$ is equal to y), so it follows that the image of Γ_1 is also a circle which contains the given four points. Since there is only one such circle, it follows that Γ_1 and its image must be the same.∎