SYNTHETIC GEOMETRY AND NUMBER SYSTEMS

When the foundations of Euclidean and non-Euclidean geometry were reformulated in the late 19^{th} and early 20^{th} centuries, the axiomatic settings did not use the primitive concepts of distance and angle measurement which are central to the exposition in the course notes (an idea which goes back to some writings of G. D. Birkhoff in the nineteen thirties). For example, the treatment in D. Hilbert's definitive *Foundations of Geometry* involved primitive concepts of *betweenness*, and *congruence of segments*, *congruence of angles* (in addition to the usual primitive concepts of lines and planes). Of course, Hilbert's approach states its axioms in terms of these concepts, and ultimately one can prove that the approach in these notes is equivalent to Hilbert's (and all other approaches for that matter). The Hilbert approach provides the setting for Greenberg's book, and Appendix B of Greenberg discusses several issues related in this approach as they apply to hyperbolic geometry. The purpose of this document is to relate Greenberg's perspective with that of the course notes. In a very lengthy Appendix we shall consider one additional aspect of nonmetric approaches to geometry; namely, finite geometrical systems.

Comparing the metric and nonmetric approaches

In Moïse, both the Hilbert and Birkhoff approaches are discussed at length, with the latter as the primary setting. As noted in comments on page 138 that book, the underlying motivation for the Birkhoff approach is that the concepts of linear and angular measurement have been central to geometry on a theoretical level since the development of algebra; as Moïse suggests, this approach did not appear in the *Elements* because the Greek mathematics at the time because the latter's grasp of algebra was extremely limited, so that even very simple algebraic issues were studied geometrically. To quote Moïse, the adoption of linear and angular measurement as undefined concepts "describe[s] the methods that in fact everybody uses."

A second advantage is that the Birkhoff approach leads to a fairly rapid development of classical geometry, which minimizes the amount of time and effort needed at the beginning to analyze the statements on betweenness and separation which may seem self-evident and possibly too simple to worry about (compare the comments in the second paragraph on page *iii* and the third paragraph on page 60 of Moïse). In a classical approach along the lines of Hilbert's development, many of the justifications for such results are not at all transparent and require long, delicate arguments which are often not helpful for understanding the big picture.

On the other hand, the modern definitions of the real number system, due to R. Dedekind (1831–1916) and G. Cantor (1845–1918) in the second half of the 19^{th} century, require some fairly sophisticated concepts which are completely outside the scope of Greek mathematics and closely related to the notion of continuity, and by a general principle of scientific thought called *Ockham's razor* (don't introduce complicated auxiliary material to explain something unless this is simply unavoidable or saves a great deal of time and effort) it is also highly desirable to look for alternative formulations which do not use the full force of the real number system's continuity properties and (in a quotation cited on page 571 of Greenberg) demonstrate that "the true essence of geometry can develop most naturally and economically."

The key to passing back and forth between the two approaches is summarized very well in the following sentence on page 573 of Greenberg:

> Every Hilbert plane [*a system satisfying all the axioms except perhaps either the Dedekind Continuity Axiom or the Euclidean parallel postulate, or possibly both*] has a field hidden in its geometry.

At the end of Section II.5 we mentioned that a similar statement is true for many abstract planes which satisfy the standard axioms of incidence and the Euclidean Parallel Postulate, and in particular this is true if the plane lies inside a 3-space satisfying the corresponding assumptions. The principle in the quotation leads directly to a four step approach to the systems he calls Hilbert planes (systems which satisfy all the axioms except perhaps either the Dedekind Continuity Axiom or the Euclidean parallel postulate, or possibly both); this approach is outlined on page 588.

REFERENCES. The approach taken in Greenberg's book is designed to be very closely connected that of the following more advanced textbook:

> **R. Hartshorne.** *Geometry: Euclid and Beyond*. Springer-Verlag, New York, 2000.

Additional background references for this material are the books by Moïse and Forder, the book by Birkhoff and Beatley, and the paper by Birkhoff; these are given in Unit II of the course notes. The latter also contain links references to many other relevant sources. Some further references for more specialized topics in Addenda A and B will be listed at the end of the latter. This will be the starting point for our discussion, and we shall begin by describing the additional features of this "hidden algebra" if the plane also satisfies Hilbert's axioms of betweenness and congruence. Much of this material appears in Greenberg, but it is dispersed throughout different sections, and it seems worthwhile to gather everything together in one place. Similarly, our discussion of non-Euclidean systems will include a chart summarizing the various places in Greenberg which deal with non-Euclidean Hilbert planes.

The level of the discussion in this document is (unavoidably) somewhat higher than that of the course notes; in many places it is probably close to the level of an introductory graduate level algebra course.

Euclidean geometry without the real number system

At the end of Section II.5 we noted that one can introduce useful algebraic coordinate systems into systems which satisfy the 3-dimensional Incidence Axioms (in Section II.1) and the Parallel Postulate (in Section II.5); since the planes of interest to us are all equivalent to planes which lie inside 3-spaces, the coordinatization result also applies to the planes that we shall consider here. If the Parallel Postulate is true, this yields algebraic coordinate structures on all system satisfying all the Hilbert axioms for Euclidean geometry except perhaps the Dedekind Continuity Axiom (see pages 134–135 and 598–599 of Greenberg). The general results of Section II.5 in the class notes state that the coordinates take values in an algebraic system called a *division ring* or a *skew-field*; informally, in such a system one can perform addition, subtraction, multiplication and division by nonzero coordinates, **but** the multiplication does not necessarily satisfy the commutative multiplication property $xy = yx$.

If one also assumes that the plane or 3-space satisfies Hilbert's axioms for betweenness and congruence (see pages 597–598 of Greenberg), then the coordinate system has additional structure called an *ordering* (special cases are described in Section 1.5 of Moïse, and the ties to geometry are discussed on pages 117–118 of Greenberg). This means that there is a subset of *positive elements* which is closed under addition and multiplication, and also has the property that for every nonzero "number" x, either x or $-x$ is positive. A standard algebraic argument shows the square of every nonzero element in an ordered division ring is positive. The coordinate system also turns out to have the *Pythagorean property* described in the following definition:

Definition. An ordered division ring K is *Pythagorean* if for each element x in the system there is a positive element y such that $y^2 = 1 + x^2$.

It is a straightforward exercise to prove the following result:

PROPOSITION. *Suppose that K is a Pythagorean field and that $a_1, \cdots, a_n$ are nonzero elements of K. Then there is unique $y \in K$ such that y is positive and*

$$y^2 = \sqrt{a_1^2 + \cdots + a_n^2} \; . \blacksquare$$

We shall be interested mainly in ordered division rings which are *ordered fields* (so that xy and yx are always equal); for the sake of completeness, we note that the books by Forder and Hartshorne describe examples of ordered division rings which are not ordered fields. Standard results in projective geometry show that the algebraic commutative law of multiplication is equivalent to a condition known as *Pappus' Hexagon Theorem*; further information on this result appears in Unit IV of the class notes and the following online document:

`http://math.ucr.edu/~res/progeom/pgnotes05.pdf`

The Pappus Hexagon Theorem is a bit complicated to state (*cf.* also Greenberg, Advanced Project 3, pp. 99–100); however, there is a more easily stated, and extremely useful, hypothesis which implies commutativity of multiplication and also considerably more:

THEOREM. *Suppose that we are given a plane or 3-space* **E** *which satisfies all the Hilbert axioms except (perhaps) the Dedekind Continuity Axiom and has coordinates given by the ordered Pythagorean division ring K. Then the following are equivalent:*

(*i*) *The plane or space* **E** *satisfies the Archimedean Continuity Axiom (see Greenberg, page* 599*).*

(*ii*) *The ordered division ring K satisfies the commutative law of multiplication and the Archimedean Property (see Greenberg, page* 601*).*$\blacksquare$

It turns out that *the conditions in* (*ii*) *hold if and only if K is order-preservingly isomorphic to a subfield of the real numbers.*$\blacksquare$

The Line-Circle and Two Circle Properties

As noted in Section III.6 of the course notes, the classical results in Euclidean geometry (including straightedge-and-compass constructions) require the two results on intersections of circle with a line or another circle named in the heading above; both are implicit in the *Elements* but not stated explicitly. The proofs of these results in the course notes rely on the fact that every positive real number has a positive square root (which is also a real number). In fact, the arguments in Section III.6 go through if we have a system with coordinates in a Pythagorean ordered field which also satisfies the commutative law of multiplication and the condition in the following definition:

Definition. A Pythagorean ordered field K is said to be **surd-complete** if every positive element of K has a positive square root in K.

In fact, the validity of the Line-Circle and Two Circle Properties turns out to be equivalent to the surd-completeness of K.

Of course, the real number system is surd-complete. Also, Sections 19.6–19.7 and 31.2 of Moïse describe a countable subfield **Surd** of the real numbers which is surd-complete and is in fact the unique minimal surd-complete subfield of the real numbers. This field also has the following basic property (not stated or used explicitly in Moïse, but closely related to the results on the "impossible" classical construction problems); it is an elementary exercise to derive this from material on field extensions in introductory graduate level algebra courses.

PROPOSITION. *Let α be a nonzero element of the surd field* **Surd**. *Then there is a unique nonzero monic polynomial $p(x)$ with rational coefficients such that the following hold:*

(i) *The surd α is a root of p.*

(ii) *If q is a nonzero polynomial with rational coefficients such that $q(\alpha) = 0$, then q is a multiple of p (hence the degree of p is minimal among all polynomials for which α is a root).*

(iii) *The degree of p is a power of* 2.■

The results in Chapter 19 of Moïse show that a classical construction problem can be done by means of straightedge and compass construction if and only if the following holds:

> *If we begin with points, lines, and circles whose defining equations only involve elements of* **Surd**, *then the defining numerical data for the constructed objects also lie in* **Surd**. — More generally, if the original data lie in an ordered field **F** which is surd-complete, then the defining numerical data for the constructed objects also lie in **F**.

We have already noted that an ordered field must be surd-complete in order to carry out the classical geometrical discussion of circles and constructions. By definition, a surd-complete field is automatically Pythagorean, and further consideration yields the following:

THEOREM. *Suppose that we are given a plane or* 3*-space* **E** *which satisfies all the Hilbert axioms except (perhaps) the Dedekind Continuity Axiom and has coordinates given by the ordered Pythagorean division ring K. Then the following are equivalent:*

(i) *The Line-Circle and Two Circle Properties are valid in the plane or space* **E**.

(ii) *The ordered division ring K is surd-complete.*■

As noted in the final paragraph on page 131 of Greenberg, it is possible to construct an Archimedean ordered field K which is Pythagorean but not surd-complete, and from this one can conclude that *it is impossible to prove the Two Circle Property from Hilbert's axioms of incidence, betweenness and congruence, even if one also assumes the coordinate field K satisfies the Archimedean Property*.■

As suggested by the discussion on pages 129–131 of Greenberg, this result implies that the very first proposition in Euclid's *Elements* (the existence of an equilateral triangle with a given line segment as one of its edges) **cannot** be proved without making some additional assumption like the Two Circle Property.

Numberless non-Euclidean geometry

A central theme in Greenberg's book is to do as much of neutral and non-Euclidean geometry as possible without using the full force of the Dedekind Continuity Axiom, and one objective of Appendix B in Greenberg is to describe coordinates in systems which satisfy all of Hilbert's axioms except perhaps the Dedekind Continuity Axiom or the Parallel Postulate (or both). Greenberg then discusses ways in which such coordinate systems can shed light on some basic questions about these geometric systems; the latter involves several ideas well beyond the advanced undergraduate level, and because of this many parts of the exposition reflect the need to be sketchy and vague about various points.

In connection with this discussion of non-Euclidean geometry without the real numbers, it seems appropriate to summarize the other locations throughout the book which discuss the consequences of assuming everything but the Parallel Postulate or Dedekind Continuity.

Page(s)	Topic(s)
129	Definition of a Hilbert plane (*i.e.*, satisfying Hilbert's axioms except possibly either the Parallel Postulate or Dedekind Continuity.
161–162	Statement that a Hilbert plane is the default setting for Chapter 4.
173	Statement of the converse to the Triangle Inequality for Hilbert planes satisfying the Two Circle Property.
175	Theorems on the role of the Parallel Postulate in a Hilbert plane.
176–191	Proofs of analogs to the results in Section V.3 of the course notes in Hilbert planes for which the Parallel Postulate does not necessarily hold; there are comments on the role of the Archimedean Property at the end, including a reference for an example of a Hilbert plane in which the angle sum of a triangle always exceeds 180°.
200	Exercise 33 discusses some issues about Hilbert planes for which the Archimedean property does not hold.
213–214	The logical equivalence (in Hilbert planes) of the Parallel Postulate and the axiom of C. Clavius (1538–1612) — namely, that parallel lines are everywhere equidistant — is discussed.
220–221	A version of an observation due to Proclus Diadochus (410–485) for Hilbert planes is stated and proved.
249–254	Proofs of analogs to the results at the beginning Section V.4 of the notes (through **AAA** congruence) are given for non-Euclidean Hilbert planes.
254–259	The behavior of parallel lines in non-Euclidean Hilbert planes (either asymptotic or having a common perpendicular) is discussed; this is closely related to the final parts of Section V.4 in the course notes.
408–409	The geometric symmetries (*automorphisms* in Greenberg) of a Hilbert plane are defined and discussed, and the significance of the Archimedean property is mentioned.
471	Exercise 69 fills in some details for the discussion on pages 408–409.
571–596	This is Appendix B.
599–601	This summarizes the axioms for geometric and algebraic systems which are central to Greenberg's book.

Final remarks

The construction of Hilbert's Field of Ends (in Part I of Appendix B) reflects the relationship between hyperbolic geometry and projective geometry that is apparent in the Beltrami-Klein model for the hyperbolic plane (see Greenberg, pages 333–346). One way of describing this relationship is described below (this requires concepts from projective geometry and can be skipped if the reader wishes to do so):

If we view the Beltrami-Klein model **Hyp** as the open unit disk in $\mathbb{R}^2$ and take the usual extension of $\mathbb{R}^2$ to the projective plane $\mathbb{RP}^2$, then the points of the latter which do not lie in **Hyp** may be viewed as points at infinity where various pencils of parallel lines in **Hyp** meet. The ideal points for asymptotically parallel lines are the points on the circle which is the boundary of **Hyp** in $\mathbb{R}^2$. Using this, it is possible to interpret some crucial properties of the real number system (arithmetic operations and order) in terms of the geometry of **Hyp**. This process can be imitated in an arbitrary Hilbert plane as follows: A general result of A. N. Whitehead shows that every 3-dimensional system satisfying the axioms of incidence and betweenness has a reasonable embedding inside a projective 3-space over an ordered division ring; for the sake of completeness we shall give the reference:

> **A. N. Whitehead.** *The Axioms of Descriptive Geometry*. Cambridge Univ. Press, New York, 1905. [The cited results appear in Chapter III. — This book is freely available on the Internet via a Google Book Search; the online address is much too long to fit on a single line, but one can get the link by doing a Google search for `whitehead axioms descriptive geometry`.]

Not surprisingly, the coordinates obtained in this manner turn out to be equivalent to the coordinates given by Hilbert's construction.

It is also possible to find reasonable embeddings of certain incidence geometries inside projective spaces even if one does not have a concept of betweenness. The following paper are basic references:

> **S. Gorn.** *On incidence geometry*. Bulletin of the American Mathematical Society **46** (1940), 158–167.

> **O. Wyler.** *Incidence geometry*. Duke Mathematical Journal **20** (1953), 601–601.

Addendum A: **Finite affine and hyperbolic planes**

At the end of Section II.1 in the course notes there is a discussion about abstract geometrical systems which are finite and satisfy the standard Incidence Axioms. Clearly one can also consider finite incidence geometries in which either the Euclidean Parallel Postulate or some negation of it holds. For example, one might assume the strongest possible negation (the **Strong** or **Universal Hyperbolic Parallel Postulate**:

> *Given a point* $\mathbf{x}$ *and a line* L *not containing* $\mathbf{x}$*, then there are at least two lines* M *and* N *through* $\mathbf{x}$ *which do not meet* L*.*

As in the main discussion above, we shall first discuss finite planes for which the Euclidean Parallel Postulate holds, and afterwards we shall discuss finite planes in which various negations of the Euclidean Parallel Postulate hold.

Finite affine planes

If we define a *finite affine plane* to be a finite (incidence) plane in which the Euclidean Parallel Postulate holds, the following two questions arise immediately:

1. Do the defining conditions yield interesting consequences? In particular, one would like to have results which are fairly simple to state but not immediately obvious from the original assumptions.
2. Do such systems arise in contexts of independent interest? (Compare the remark by J. L. Coolidge, quoted on page 35 of the course notes, in document `geometrynotes2a.pdf`: *The unproved postulates ... must be consistent, but they had better lead to something interesting*.)

The following simple result suggests an affirmative answer to the first question:

THEOREM. *Let* $(\mathbf{P}, \mathfrak{L})$ *be a finite incidence plane. Then the following hold:*

(i) *The number of points in* $\mathbf{P}$ *is a perfect square (which must be at least* 4 *since* $\mathbf{P}$ *has at least three points).*

(ii) *If the positive integer* $n \geq 2$ *is such that* $\mathbf{P}$ *has* n^2 *points, then every line in* P *contains exactly* n *points, and every point in* P *lies on exactly* $(n+1)$ *lines.*

A reference for this result is Exercise 7 on page 33 of the following document:

`http://math.ucr.edu/~res/progeom/pgnotes04.pdf`

If $\mathbb{GF}(n)$ is a finite field with n elements (for example, we can take $\mathbb{GF}(p)$ to be $\mathbb{Z}_p$ if p is a prime), then the coordinate plane $\mathbb{GF}(n)^2$ with the usual lines (namely, all subsets of the form $\mathbf{x} + V$ where $\mathbf{x}$ is an arbitrary vector and V is an arbitrary 1-dimensional vector subspace) is an affine plane with n^2 elements; standard results from (graduate level) abstract algebra courses imply that such fields exist if and only if n is a prime power.

In many important respects, affine geometry — and particularly finite affine geometry — is best viewed as part os projective geometry (see Unit IV of the notes), and in particular the usual approach to finite affine planes is to construct associated finite projective planes using the methods of pages 47–48 and 62–65 of the following document:

`http://math.ucr.edu/~res/progeom/pgnotes03.pdf`

Finite projective planes have been studied extensively and effectively during the past 100 years or so, and they turn out to have important practical uses in mathematical statistics, especially in the theory of experimental design. Further discussion and references are given on pages 37–39 of the course notes and on pages 79–82 and 84–86 of the previously cited online document ... `pg-notes04.pdf`; the book by Bose in the bibliography is an important reference for the applications of finite projective planes and related structures.

Finite hyperbolic planes

Since there is a fairly extensive theory of finite affine and projective planes, it is natural to speculate about finite analogs of hyperbolic planes. This topic has beens studied sporadically and only to a limited extent, and the literature is somewhat scattered. Therefore the following summary almost surely overlooks some work on this question.

The most naïve and obvious approach to defining a finite non-Euclidean plane is to say it is a finite plane which does not satisfy the Euclidean Parallel Postulate. However, as in the preceding discussion of finite affine planes, there are immediate questions regarding the logical consequences of such a definition or the existence of models which are relevant to questions of independent interest. It is possible to go even further and ask whether the given definition is enough by itself to yield structures which are worth studying in some degree of detail, but we shall not try to address this question because it gets into subjective (but nevertheless important!) considerations.

BASIC CONSEQUENCES AND EXAMPLES. We shall begin by deriving a simple but noteworthy property of non-Euclidean planes:

PROPOSITION. *Let $(\mathbf{P}, \mathfrak{L})$ be a finite incidence plane in which there is a line L and a point $C \notin L$ such that there are at least two parallels to L through C. Then P contains (at least) four points, no three of which are collinear.*

Proof. Let A and B be two points of L, and let C be as above. Choose D and E such that CD and CE are distinct lines which are parallel to $L = AB$ (such lines exist by the hypothesis). We claim that the points A, B, C, D, E are distinct; certainly the first three are because $C \notin AB$, while the conditions on D and E also imply that neither of these points can be A, B or C, and $D \neq E$ because $CD \neq CE$. It will suffice to prove that $\{A, B, C, D, E\}$ contains four points, no three of which are collinear.

There are exactly 10 subsets of $\{A, B, C, D, E\}$ which contain exactly 3 elements, and they may be listed as follows:

$$\begin{gathered}
\{A, B, C\} \\
\{A, B, D\} \\
\{A, B, E\} \\
\{A, C, D\} \\
\{A, C, E\} \\
\{A, D, E\} \\
\{B, C, D\} \\
\{B, C, E\} \\
\{B, D, E\} \\
\{C, D, E\}
\end{gathered}$$

Since $C \notin L = AB$, we know that $\{A, B, C\}$ is noncollinear. Also, both $\{A, B, D\}$ and $\{A, B, E\}$ are noncollinear because $AB \cap CD = AB \cap CE = \emptyset$ by assumption. Similarly, the sets $\{A, C, D\}$ and $\{A, C, E\}$ are not collinear for the same reason, and likewise for $\{B, C, D\}$ and $\{B, C, E\}$. In a different direction, $CD \neq CE$ implies that $\{C, D, E\}$ must be noncollinear. Therefore, by process of elimination we see that *the* **only** *three point subsets of* $\{A, B, C, D, E\}$ *which* **might be** *collinear are* $\{A, D, E\}$ *and* $\{B, D, E\}$.

Since neither of these subsets is contained in $\{A, B, C, D\}$ or $\{A, B, C, E\}$, it follows that each of the latter is a subset of $\mathbf{P}$ containing four points, no three of which are collinear.■

Note. It is easy to construct examples of (finite) planes which do not contain a subset of four or more noncollinear points such that *every* subset of three points is collinear (if every subset of three points in $\mathbf{X} \subset \mathbf{P}$ is collinear, it is a straightforward exercise to prove that $\mathbf{X}$ is collinear). The most obvious example of this sort is a plane with three points such that the lines are all subsets containing exactly two points, but we can also construct examples with any finite number of points as follows: Given an integer $n \geq 3$, let $\mathbf{P} = \{0, 1, \cdots, n\}$ and take $\mathfrak{L}$ to be the family of all subsets $\{0, k\}$ where $k > 0$ together with $\{1, \cdots, n\}$. It is then a routine exercise to verify that $(\mathbf{P}, \mathfrak{L})$ is an incidence plane, and since every subset with four or more points must contain at least three points in $\{1, \cdots, n\}$, it follows that if $\mathbf{X}$ is a (noncollinear) subset of $\mathbf{P}$ containing 4 or more points, then there is a collinear subset of $\mathbf{X}$ which contains 3 points.■

We have already raised questions whether the negation of the Euclidean Parallel Postulate is a strong enough assumption to yield a significant body of noteworthy results, and we have suggested the option of assuming the Universal Hyperbolic Parallel Postulate. Before doing so, we shall give examples of finite non-Euclidean planes which do not satisfy the Universal Hyperbolic Parallel Postulate. The basic idea is simple; namely, we take an affine plane $(\mathbf{P}, \mathfrak{L})$ in which all lines have at least three points, and we remove a line L_0 from $\mathbf{P}$.

> Formally, if $(\mathbf{P}, \mathfrak{L})$ is an affine incidence plane as above, let L_0 be a line in $\mathbf{P}$, and set $\mathbf{Q}$ equal to $\mathbf{P} - L_0$. Now let M_0 be a second line in $\mathbf{P}$ which meets L_0 ar some point $\mathbf{z}_0$. If $\mathbf{z}_1$ and $\mathbf{z}_2$ are two distinct points of M_0 other than $|bfz_0$, then $\mathbf{z}_1$ and $\mathbf{z}_2$ lie on distinct lines of L_1 and L_2 in $\mathbf{P}$ such that $L_1 || L_0$ and $L_2 || L_0$, and we claim that $L_2 \cap \mathbf{Q}$ is the unique line in $\mathbf{Q}$ which contains bfz_2 and is disjoint from $L_1 \cap \mathbf{Q}$. It follows immediately that $\mathbf{z}_2 \in L_2 \cap \mathbf{Q}$ (if not, then $\mathbf{z}_2 \in L_0$, so that $\mathbf{z}_2 \in L_0 \cap M = \{\mathbf{z}_0\}$, contradicting $\mathbf{z}_2 \neq \mathbf{z}_0$), and clearly $L_1 \cap \mathbf{Q} || L_2 \cap \mathbf{Q}$ because $L_1 || L_2$. To prove uniqueness, we must show that if K is a line in $\mathbf{Q}$ such that $\mathbf{z}_2 \in K$ and $K || L_1 \cap \mathbf{Q}$, then $K = L_2 \cap \mathbf{Q}$.
>
> Suppose that $K = K' capQ$ is such that $\mathbf{z}_2 \in K$ and $K || L_1 \cap \mathbf{Q}$. If $K' \cap L_1 = 0$, then $K' = L_2$ because $\mathbf{P}$ is affine, so that $K = L_2 \cap \mathbf{Q}$. On the other hand, if $K' \cap L_1 \neq \emptyset$, then the intersection must lie in L_0. But this implies that $L_1 \cap L_0$ is also nonempty, contradicting the choice of L_1 as a line parallel to L_0. This proves that $K' = L_2$ and $K = L_2 \cap \mathbf{Q}$.
>
> Now let $\mathbf{w}_0$ be a second point of L_0, let M_1 be the unique parallel to M_0 through $\mathbf{w}_0$, and let $\mathbf{w}_1$ be a second point on M_1, so that $M_1 \cap L_0 = \{\mathbf{w}_0\}$. Then $\mathbf{Q} \cap M_1$ and $\mathbf{Q} \cap \mathbf{z}_0\mathbf{w}_1$ are two lines in $\mathbf{Q}$ which pass through $\mathbf{w}_1$ and do not meet $\mathbf{Q} \cap M_0$.■

THE MINIMALITY PROPERTY. Even if we assume the Universal Hyperbolic Parallel Postulate, there are some "bloated" examples that fit the formal criteria but are really "too big" to be thought of as planes. For example, if $(\mathbf{S}, \mathfrak{L}, \mathfrak{P})$ is an affine 3-space, then at least formally we can make $\mathbf{S}$ into an incidence plane by simply decreeing that $\mathbf{S}$ is a plane and ignoring the family $\mathfrak{P}$. It follows immediately that $(\mathbf{S}, \mathfrak{L})$ is an incidence plane.

CLAIM. *The system* $(\mathbf{S}, \mathfrak{L})$ *satisfies the Universal Hyperbolic Parallel Postulate.*

Proof of Claim. Let L be a line and let $\mathbf{x}$ be a point of $\mathbf{S}$ not on L. Then there is a unique parallel M to L through $\mathbf{x}$ in the affine 3-space $(\mathbf{S}, \mathfrak{L}, \mathfrak{P})$. Let $Q \in \mathfrak{P}$ be such that $L \subset Q$ and

$\mathbf{x} \in Q$. Since Q is a proper subset of $\mathbf{S}$, we can find some point $\mathbf{y}$ in $\mathbf{S}$ which does not lie in Q. It will suffice to prove that $\mathbf{xy}$ and L have no points in common. If there was some point $\mathbf{z}$ on both, then $\mathbf{x}, \mathbf{z} \in Q$ would imply that the line joining them — which is $\mathbf{xy}$ — would also lie in Q, contradicting our choice of $\mathbf{y}$. Therefore $\mathbf{xy}$ and M are two lines through $\mathbf{x}$ which are disjoint from L.■

Still more examples of this type can be constructed by letting $\mathbf{P} = \mathbb{F}^n$, where $\mathbb{F}$ is a finite field and $n \geq 4$. As in the preceding examples, these systems have higher dimensional incidence structures that are described on pages 31–36 of the following document:

`http://math.ucr.edu/~res/progeom/pgnotes02.pdf`

Clearly we have turned higher dimensional incidence structures into planes by the formal trick of simply ignoring all higher dimensional structure. One way to avoid such questionable constructions is to assume an additional property. It will be convenient to formulate this in terms of an auxiliary concept:

Definition. Let $(\mathbf{P}, \mathfrak{L})$ be an incidence plane. A subset $Q \subset \mathbf{P}$ is *flat* if for each pair of distinct points $\mathbf{x} \neq \mathbf{y}$ in Q, the line joining them is contained in Q. There is an obvious close relationship between this condition and one of the 3-dimensional incidence axioms.

Definition. An incidence plane $(\mathbf{P}, \mathfrak{L})$ is said to be **minimal** or **irreducible** provided the only noncollinear flat subset of $\mathbf{P}$ is $\mathbf{P}$ itself. We shall say that $(\mathbf{P}, \mathfrak{L})$ is **reducible** if this condition does not hold.

Clearly all of the "bloated" examples are reducible; in fact, if we are given three noncollinear points in one of them, then the "plane" containing them (in the sense of the full incidence structure) is a proper, noncollinear, flat subset.

Before proceeding, we should note that, with one exception, all finite affine planes are irreducible and, with no exceptions, all finite projective planes are irreducible.

THEOREM. *Let $(\mathbf{P}, \mathfrak{L})$ be a finite incidence plane which is either a projective plane or an affine plane with more than 4 points. Then $(\mathbf{P}, \mathfrak{L})$ is irreducible.*

It is fairly straightforward to show that two affine planes with exactly 4 points have isomorphic incidence structures (the lines are the two point subsets in this case), and if we combine this with the conclusion of the theorem we see that, up to incidence isomorphism, *there is exactly one affine incidence plane which is reducible (the 4 point model turns out to be reducible*, for every line contains exactly two points, and therefore **every** *subset of this model is flat* in the sense of the definition).

Proof. Clearly there are two cases, depending upon whether the plane is projective or affine. We recall that a projective plane is one such that every pair of lines has a point in common, and every line contains at least three points. Some basic properties of such objects are established in the previously cited document ... `pgnotes04.pdf`.

Suppose first that $(\mathbf{P}, \mathfrak{L})$ is projective, and let Q be a flat, noncollinear subset of $\mathbf{P}$. Let A, B, C be noncollinear points in Q, and let $X \in \mathbf{P}$. If $X \in AB$ or $X \in BC$ then by flatness we know that $X \in Q$, so suppose that X lies on neither line. It follows that the line XC is distinct from the line AB, and hence it meets the latter in some point D; the points C and D are distinct, for otherwise the two distinct points B and $C = D$ would lie on the two distinct lines AB and AC. We know that $D \in Q$ because $D \in AB$, and therefore we can conclude that the line $CD = XC$ is contained in Q. But this means that $X \in Q$; therefore we have shown that Q contains every point of P, and hence we have $Q = \mathbf{P}$.

The preceding argument also works in the affine case provided the lines AB and CX have a point in common, and hence in affine case we can say that a point $X \in \mathbf{P}$ lies in Q except perhaps if X lies on the unique line L through C such that $AB||L$. If we switch the roles of A and C in this argument, we also see that a point $X \in \mathbf{P}$ lies in Q except perhaps if X lies on the unique line M through A such that $BC||L$. Combining these, we see that a point $X \in \mathbf{P}$ lies in Q except perhaps if $X \in L \cap M$. The lines L and M are distinct because one is parallel to AB and the other contains a point of AB, and therefore we have shown that Q contains all but at most one point of $L \cap M$. We should note that these two lines do have a point in common, for if they did not then both lines would be parallel to the nonparallel, nonidentical lines AB and BC, and this is impossible in an affine plane. We shall denote this point by E.

At this step of the argument we shall finally use the assumption that each line in $\mathbf{P}$ contains more than two points. The point E is the only point of $\mathbf{P}$ which might not be in Q. We know that $E \in L$, so it will suffice to show that at least two points of L are contained in Q. By construction, we have $C \in L \cap Q$, and the assumption on the order of $\mathbf{P}$ implies that there is a point $Y \in L$ such that $Y \neq X, C$. Our reasoning thus far implies that $Y \in Q$, and therefore the flatness assumption implies that the entire line L is contained in Q; therefore $E \in Q$, so we have shown that every point of $\mathbf{P}$ lies in Q.■

Drawings to accompany this proof are posted in the following file:

`http://math.ucr.edu/~res/math133/irreducibleplanes1.pdf`

To motivate the concept of irreducibility further, we shall also sketch a proof that

every Hilbert plane is also irreducible.

In fact, all that one needs to prove irreducibility are the Axioms of Incidence and Order (Betweenness and Plane Separation). An illustrated proof is given in the following online document:

`http://math.ucr.edu/~res/math133/irreducibleplanes2.pdf`

In view of the preceding observations, we shall define a **finite** (synthetic) **hyperbolic plane** to be an incidence plane which is irreducible and satisfies the Universal Hyperbolic Parallel Postulate. One example (in fact, the smallest possible such system) satisfying these conditions is given in the extremely readable paper by L. M. Graves which is cited in the Bibliography. This model contains 13 points and 26 lines.

Additional examples, often satisfying stronger versions of the Universal Hyperbolic Parallel Postulate (specifically, how many parallels exist through a given point) and other desirably geometric conditions (for example, symmetry properties), appear in numerous other articles, including the 1963 paper by R. Sandler, the 1964 and 1965 papers by D. W. Crowe, the 1965 paper by M. Henderson, and the 1970 paper by S. H. Heath. Still further results along these lines appear in the 1971 paper by R. Bumcrot, which also establishes some restrictions on the numerical data of finite hyperbolic planes.

Examples like the preceding ones are informative in several respects, but ultimately one would like examples which are relevant to other geometrical topics of independent interest. In many cases, the outside interest arises from properties of the Beltrami-Klein model and the role of hyperbolic geometry in a setting of F. Klein (the *Erlangen Program*), which was designed to provide a unified framework for the various types of geometry that existed when it was formulated in 1870–1872. Here is an online reference describing Klein's influential views and their impact, followed by a link to an English translation of Klein's original paper:

`http://en.wikipedia.org/wiki/Erlangen_program` http://www.ucr.edu/home/baez/erlangen/erlangen_tex.pdf

The place of hyperbolic geometry in this organizational scheme is easy to describe. Namely, the Beltrami-Klein model for hyperbolic geometry provides the basis for integrating hyperbolic geometry into Klein's framework.

Some very brief papers around 1940 suggested that there might not be any finite analogs of the classical hyperbolic plane aside from some trivial ones. The subsequent 1946 paper by R. Baer was another early (and discouraging) step in the search for "extrinsically motivated" examples of finite hyperbolic planes. On the other hand, the 1962 paper by T. G. Ostrom produced examples similar to Graves' which in many respects reflected the role of classical hyperbolic geometry in Klein's Erlangen Program; a crucial link between Ostrom's paper and Klein's viewpoint is studied in the 1955 paper by B. Segre. Other articles in this direction include the 1965 and 1966 papers by Crowe, the 1966 paper by R. Artzy, and the 1969 paper by G. I. Podol'nyĭ; the highly symmetric examples of finite hyperbolic planes in previously cited articles are also closely related to Klein's Erlangen Program.

We could go into greater detail about results from the individual articles listed below, but to keep the discussion relatively brief we shall merely conclude by mentioning the 1977 survey by J. Di Paola, which discusses finite hyperbolic planes in the more general context of finite geometries.

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ADDENDUM B: **Classical geometry and topological spaces**

Another alternate approach to the foundations of geometry is to characterize them using the theory of topological spaces, which is fundamental to much of modern mathematics. There have been several studies in this direction, but we shall only mention one pair of papers in which the necessary mathematical background does not go beyond topics covered in standard undergraduate courses for mathematics majors.

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