SOLUTIONS TO ADDITIONAL EXERCISES FOR III.1 AND III.2

Here are the solutions to the additional exercises in `perpexercises.pdf`. Illustrations to accompany these solutions are given in the online file

`perpfigures.pdf`

in the course directory.

D1. We shall follow the hints. Take a basis B for V (which has r elements) and extend it to a basis for $\mathbb{R}^n$ by adding a suitable set of $n-r$ vectors A; order the basis so that the elements of B come first. If we apply Gram-Schmidt process to obtain an orthonormal basis C of $\mathbb{R}^n$ from $B \cup A$, then by construction the first r vectors in C will form an orthonormal basis for V. Let A' be the last $n-r$ vectors in C; we claim that A' forms an orthonormal basis for $V^\perp$.

First of all, every vector in A' lies in $V^\perp$, for every vector in V has the form $\sum_{j \leq r} t_j \mathbf{c}_j$ and the dot product of such a vector with $\mathbf{c}_k$ is zero if $k > r$. Therefore $V^\perp$ contains the $(n-r)-$dimensional vector subspace spanned by A'. To see that nothing else can be contained in $V^\perp$, consider a vector $\mathbf{y}$ which is not a linear combination of the vectors in A'. Since C is an orthonormal basis, we must have $\mathbf{y} = \sum_{j \leq n} t_k \mathbf{c}_j$ where $t_m \neq 0$ for some $m \leq k$. But the latter implies that $\mathbf{y} \cdot \mathbf{c}_m = t_m \neq 0$, and therefore $\mathbf{y}$ cannot lie in $V^\perp$. Thus the vectors in A' form a basis of this subspace and hence its dimension is $n-r$.

To conclude, as noted in the hint it suffices to prove that V is a vector subspace of $\left(V^\perp\right)^\perp = V$ and the dimensions of these two subspaces are equal. The first statement follows since $\mathbf{v} \in V$ implies $\mathbf{v} \cdot \mathbf{x} = 0$ for all $\mathbf{x} \in V^\perp$, and the first follows because the dimension of $\left(V^\perp\right)^\perp = V$ is equal to

$$n \ - \ \dim V^\perp \ = \ n \ - \ (n-r) \ = \ r \ = \ \dim V \ .$$

Since $V_1 \subset V_2$ and $\dim V_1 = \dim V_2$ imply $V_1 = V_2$, the equality of V and $\left(V^\perp\right)^\perp = V$ follows immediately.■

Note. One important consequence of the preceding exercise is the following: *If V and W are vector subspaces of $\mathbb{R}^n$ such that $V \neq W$, then $V^\perp \neq W^\perp$.* — For if $V^\perp = W^\perp$, then their orthogonal complements, which by the exercise are V and W respectively, would also have to be equal.

D2. By the preceding exercise we know that $\dim V^\perp = 2$ and $\dim W^\perp = 1$. Furthermore, since V and $W^\perp$ are distinct $1-$ dimensional subspaces, it follows that the dimension of their intersection is strictly less than 1 and hence the intersection must be $\{\mathbf{0}\}$.

Since V and $W^\perp$ are distinct $1-$dimensional vector subspaces, it follows that their orthogonal complements $V^\perp$ and $\left(W^\perp\right)^\perp = W$ are distinct $2-$dimensional vector subspaces (see the note following the solution of D1). Therefore the linear sum $V^\perp + W$ properly contains each of them (otherwise they would be equal), so its dimension is at least 3; since we are in $\mathbb{R}^3$, the dimension must be exactly 3 and the linear sum is just $\mathbb{R}^3$. Applying the Dimension Formula we see that

$$\dim W \cap V^{\perp} \quad = \quad \dim W \; + \; \dim V^{\perp} \; - \; \dim \mathbb{R}^3 \quad = \quad 2 \; + \; 2 \; - \; 3 \quad = \quad 1$$

D3. Write the line and plane as $\mathbf{x} + V$ and $\mathbf{x} + W$ respectively; the assumptions imply that V is not equal to $W^{\perp}$ and hence $M = \mathbf{x} + \left(W \cap V^{\perp}\right)$ is a line which is contained in both $P = \mathbf{x} + W$ and in the plane $Q = \mathbf{x} + V^{\perp}$ Since Q is the unique plane through $\mathbf{x}$ which is perpendicular to L, it follows that M has the properties described in the statement of the exercise.

To see that there is only one line, suppose that M' has the required properties. Then it follows that $M' \subset Q$, and since $M' \subset P$ is assumed we know that M' is contained in $P \cap Q$; since the latter is a line, it follows that we have $M' = P \cap Q$, and since the intersection is M we have $M' = M$.■

D4. The condition $a < 2x$ follows from the Triangle Inequality for triples of non-collinear points. Conversely, if we have $a < 2x$, then we also have

$$0 \quad < \quad h \quad = \quad \sqrt{x^2 - \frac{a^2}{4}} \; .$$

By the Protractor and Ruler Postulates we can construct a right triangle ΔABC such that $AB \perp BC$, $d(A, B) = a/2$, and $d(B, C) = h$. By the Pythagorean Theorem we know that $d(A, C) = 90°$. Now take $D \in (AB$ such that $d(A, D) = a$. It then follows that $d(B, D) = a/2$ and by **SAS** and perpendicularity we have $\Delta ABC \cong \Delta DBC$. It follows that $d(D, C) = d(A, C) = x$, and therefore the triangle ΔABC is an isosceles triangle such that the lengths of two sides are equal to x and the length of the third side is equal to a.■