An example involving arc length

Here is an example in which the differentiation and antidifferentiaion can be handled efficiently using the methods of single variable calculus classes:

Problem. *Let*

$$\mathbf{x}(t) \quad = \quad \left((t+1)^{3/2}, (t+2)^{3/2} \right)$$

for $t \geq 0$*. Find the arc length* $s(t)$ *of the curve from time* 0 *to time* t*, and express* t *as a function of the arclength* s*.*

Solution. First use the arc length formula to find $s(t)$:

$$s(t) \quad = \quad \int_0^t \sqrt{x_1'(u)^2 + x_2'(u)^2}\, du \quad = \quad \int_0^t \sqrt{\left(\tfrac{3}{2}\sqrt{u+1}\right)^2 + \left(\tfrac{3}{2}\sqrt{u+2}\right)^2}\, du \quad =$$

$$\frac{3}{2} \int_0^t \sqrt{2u+3}\, du \quad = \quad \frac{3}{2}\,\frac{1}{2}\,\tfrac{2}{3}(2u+3)^{3/2}\Big|_0^t \quad = \quad \frac{1}{2}\left[(2t+3)^{3/2} - 3^{3/2}\right] .$$

To express t in terms of s solve this equation for s.

$$s \quad = \quad \frac{1}{2}\left[(2t+3)^{3/2} - 3^{3/2}\right] \quad \Longrightarrow \quad 2s \quad = \quad (2t+3)^{3/2} - 3^{3/2} \quad \Longrightarrow$$

$$2s + 3^{3/2} \quad = \quad (2t+3)^{3/2} \quad \Longrightarrow \quad \left(2s + 3^{3/2}\right)^{2/3} \quad = \quad 2t+3 \quad \Longrightarrow$$

$$t \quad = \quad \frac{1}{2}\left[\left(2s + 3^{3/2}\right)^{2/3} \; - \; 3\right] .$$