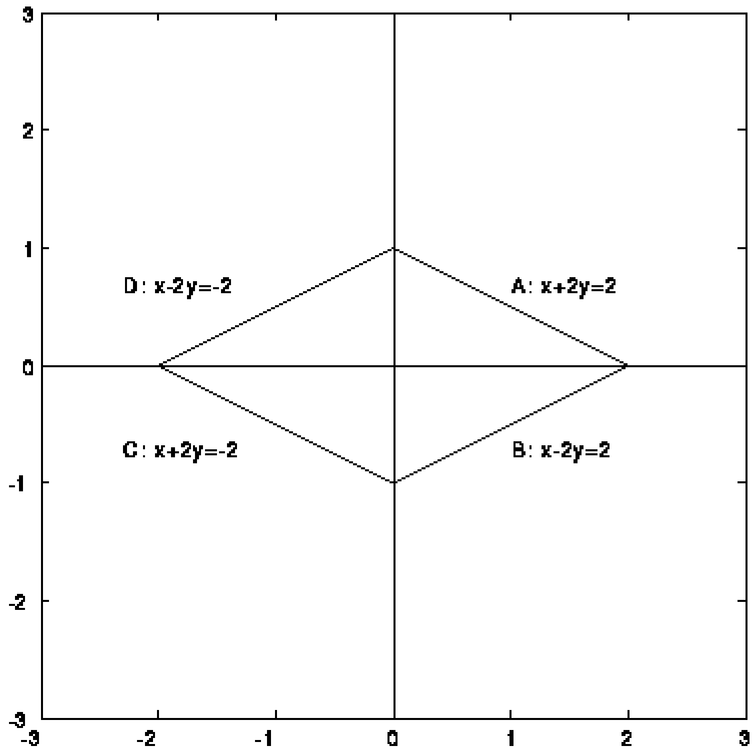
Examples for change of variables transformations

Linear transformations. Here is a typical example:



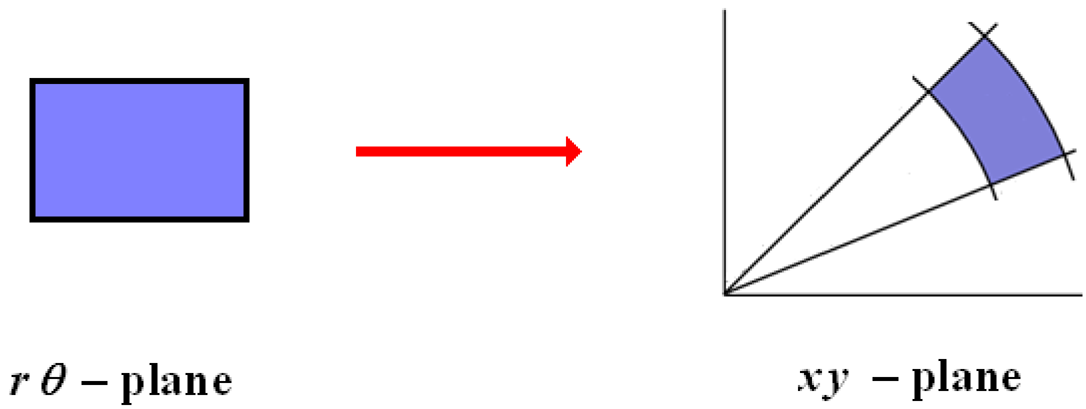
(**Source:** http://www.math.oregonstate.edu/home/programs/undergrad/CalculusQuestStudyGuides/vcalc/change/change.html)

If we let **T** be the linear transformation $(u, v) = \mathbf{T}(x, y) = (x + 2y, x - 2y)$, then **T** maps the diamond shaped region in the xy – plane bounded by the four lines

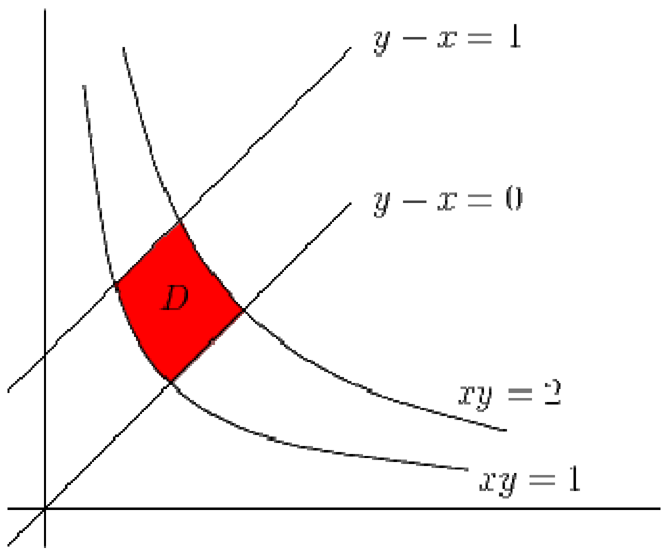
$$x + 2y = 2, \quad x - 2y = 2, \quad x + 2y = -2, \quad x - 2y = -2$$

to the square shaped region in the uv – plane bounded by the lines $u = \pm 2$ and $v = \pm 2$. More generally, linear transformations of the form $\mathbf{T}(x, y) = (ax + cy, bx + dy)$ with $ad - bc \neq 0$ always send parallelograms in the domain (source) to parallelograms in the codomain (target); in this context we are thinking of a rectangle as a special case of a parallelogram. If we consider the ***inverse (linear) transformation*** **S** which is defined as usual by $\mathbf{S}(u, v) = (x, y)$ if and only if $(u, v) = \mathbf{T}(x, y)$, then S will map the square bounded by the lines $u = \pm 2$ and $v = \pm 2$ to the diamond shaped region depicted above. As usual, we can give explicit formulas for u and v by solving the system of linear equations $u = x + 2y$ and $v = x - 2y$. The solution is given by $x = (u + v)/2$ and $y = (u - v)/4$. Note that the Jacobian of **T** is constant and equal to -4.

Nonlinear transformations. In the plane, the most basic example of a nonlinear transformation is the standard polar coordinates transformation $(x, y) = \mathbf{P}(r, \theta) = (r \cos \theta, r \sin \theta)$. Recall that this transformation maps a rectangle (in the $r\theta$ – plane) to a region in the xy – plane bounded by two concentric circles and two lines as in the illustration below. Multivariable calculus texts do a great deal of work with this transformation.



Here is another example of a nonlinear transformation:



(**Source:** **http://www.math.umn.edu/~nykamp/m2374/readings/changevardintex/changevardintex18x.png**)

If we apply the transformation $(u, v) = \mathbf{T}(x, y) = (y - x, xy)$, then $\mathbf{T}$ maps the red region D in the xy – plane into the rectangular region of the uv – plane given by $0 \le u \le 1$ and $1 \le v \le 2$. Inverse transformations to $\mathbf{T}$ are given by the following formula:

$$(x, y) = \tfrac{1}{2}\left(u \pm \mathrm{SQRT}(u^2 + 4v), -u \pm \mathrm{SQRT}(u^2 + 4v)\right)$$

Here is a derivation of the displayed equation: Since $u = y - x$ it follows that $y = u + x$, so that we have $v = xy = ux + x^2$. Rewriting this as $0 = x^2 + ux - v$ we can use the quadratic formula to solve for x, and from this we can also solve for y using the first equation $u = y - x$.

Note that the Jacobian of $\mathbf{T}$ at (x, y) is equal to $-(x + y)$, and furthermore the Jacobians of the inverse transformations are equal to $\pm\,\mathrm{SQRT}(u^2 + 4v)$.

Another nonlinear example: **http://www.math.umn.edu/~rogness/multivar/nonlineartransformation.html**