CURVE CONGRUENCE

Recall a congruence transformation on $\mathbb{R}^3$ is a 1-1 onto map $\mathbb{R}^3 \to \mathbb{R}^3$ of the form $\Phi(X) = AX + C$

X: 3×1 column vector
A: orthogonal 3×3 matrix (cols. orthonormal) and $\det A = +1$
C: constant 3×1 column vector

Theorem Let α & β be two regular smooth curves defined on $(s_0 - h, s_0 + h)$ with "arc length like" parametrizations ($|\alpha'| = |\beta'| = 1$). Then there is a congruence transformation Φ such that $\beta(s) = \Phi(\alpha(s))$ if and only if the curvature and torsion functions are equal, PROVIDED the curvatures are positive at s_0.

We need this to define Frenet trihedra

DERIVATION. Suppose $\beta(s) = \Phi(\alpha(s))$ where Φ is a congruence transformation, and write $\Phi(X) = AX + C$ as above, so that

$\beta = A\alpha + C$. We then have

$\beta' = A\alpha'$, so that the unit tangents $T_\gamma = \gamma'$ (in this case) satisfy $T_\beta = A \cdot T_\alpha$.

Since $T_\beta' = \kappa_\beta N_\beta$ we have

$$T_\beta' = \kappa_\beta N_\beta = (A \cdot T_\alpha)' = A \cdot T_\alpha' = A(\kappa_\alpha N_\alpha) = \kappa_\alpha (A N_\alpha).$$

But if A is orthogonal then $|AY| = |Y|$ for all vectors Y, so that $\kappa_\beta = |T_\beta'| = |AT_\alpha'| = |T_\alpha'| = \kappa_\alpha$, showing that the curvatures agree.

Next, we claim that A orthogonal and $\det A = +1$ imply $A(X \times Y) = AX \times AY$ for X, Y orthonormal. Now AX and AY are unit vectors and perpendicular, and $A(X \times Y)$ is a unit vector perpendicular to both, so it must be $\pm AX \times AY$. Now

3×3 matrix written in column form

$$\det[AX, AY, AX \times AY] = 1$$

So the correct sign is given by

$$\det[AX, AY, A(X \times Y)]$$

But now $(AX, AY, A(X \times Y)) =$

$$A\,(X, Y, X \times Y) \Longrightarrow$$

determinant is $\underbrace{\det A}_{\substack{\parallel \\ 1 \text{ by assumption}}} \cdot \underbrace{\det(X, Y, X \times Y)}_{\substack{\parallel \\ |X \times Y|^2 = 1}}$

So that $AX \times AY = A(X \times Y)$.

Let's apply this to T_γ and N_γ where $\gamma = \alpha$ or $\Phi\alpha = \beta$. We have

$$B_\beta = T_\beta \times N_\beta = AT_\alpha \times AN_\alpha = AB_\alpha.$$

We use this to compare the torsions τ_α, τ_β.

$$B'_\beta = -\tau_\beta N_\beta$$

$$B'_\beta = (AB_\alpha)' = AB'_\alpha = -A(\tau_\alpha N_\alpha) =$$

$$-\tau_\alpha AN_\alpha = -\tau_\alpha N_\beta.$$ Since $N_\beta \neq 0$

this implies that $-\tau_\alpha = -\tau_\beta$, or $\tau_\alpha = \tau_\beta$.

Hence $\beta = \Phi\alpha \Rightarrow$ curvature and torsion = when curvatures nonzero.

CONVERSELY Suppose that we have β such that $\tau_\beta = \tau_\alpha$ and $\kappa_\beta = \kappa_\alpha$ (nonzero!).

We need to find Φ such that $\beta(s) = \Phi \circ \alpha(s)$.

Let F_β and F_α be the matrices whose columns are the Frenet trihedra for ~~?~~ β and α.

<u>CLAIM</u> There is an orthogonal matrix A s.t. $\det A = +1$ and $A F_\alpha(s_0) = F_\beta(s_0)$.

Let $A = F_\beta F_\alpha^{-1}$; this matrix is orthogonal since a product of orthogonal matrices is orthogonal, and $\det A = (\det F_\beta)(\det F_\alpha^{-1}) = (\det F_\beta)(\det F_\alpha)^{-1} = 1 \cdot 1 = 1$.

$\boxed{\det F_\gamma \equiv 1}$

Now let $\left\{ \begin{array}{l} C = \beta(s_0) - A\alpha(s_0) \\ \Phi(X) = AX + C \end{array} \right\}$ Then $\Phi\alpha(s_0) =$

$$A\alpha(s_0) + C = A\alpha(s_0) + (\beta(s_0) - A\alpha(s_0)) = \beta(s_0).$$

Let $\gamma = \Phi\alpha$. We want to show $\gamma = \beta$.

This require proving $\underbrace{\Phi(\alpha(s_0))}_{\gamma(s_0)=} = \beta(s_0)$ [DONE], and $F_\gamma(s_0) = F_\beta(s_0)$, or equivalently

$$T_\gamma(s_0) = T_\beta(s_0),\ N_\gamma(s_0) = N_\beta(s_0),\ B_\gamma(s_0) = B_\beta(s_0)$$

If we know these, then $\tau_\alpha = \tau_\beta \Rightarrow$
$\tau_\gamma = \tau_\alpha = \tau_\beta$ (by the first part of the thm.),
and similarly $\kappa_\alpha = \kappa_\beta \Rightarrow \kappa_\gamma = \kappa_\alpha = \kappa_\beta$,
so the uniqueness part of the Fundamental Theorem of Curve Theory implies $\beta = \gamma = \Phi\alpha$

By construction we know that
$T_\beta(s_0) = A\,T_\alpha(s_0)$, $N_\beta(s_0) = A\,N_\alpha(s_0)$, $B_\beta(s_0) = A\,B_\alpha(s_0)$,
so we need only prove similar eqns with γ replacing β.

But $T_\gamma(s_0) = \frac{d}{ds}(A\alpha + C)_{s_0} = A\alpha'(s_0) = AT_\alpha(s_0)$,

and $N_\gamma(s_0) = \frac{1}{\kappa_\gamma(s_0)} T_\gamma'(s_0) = \frac{1}{\kappa_\gamma(s_0)} \frac{d}{ds}(AT_\alpha)_{s_0} =$

$\frac{1}{\kappa_\gamma(s_0)} A \cdot T_\alpha'(s_0) = \frac{1}{\kappa_\gamma(s_0)} A\,\kappa_\alpha(s_0) N_\alpha(s_0) =$

$\frac{\cancel{\kappa_\gamma(s_0)}}{\cancel{\kappa_\alpha(s_0)}} AN_\alpha(s_0)$, so $\left\{ \begin{array}{l} T_\gamma(s_0) = T_\beta(s_0) \\ N_\gamma(s_0) = N_\beta(s_0). \end{array} \right.$

(1 since $\kappa_\gamma = \kappa_\alpha$)

Finally,

$$B_\gamma(s_0) = T_\gamma(s_0) \times N_\gamma(s_0) \underset{\text{PREVIOUS}}{=\!=}$$

$$T_\beta(s_0) \times N_\beta(s_0) = B_\beta(s_0).$$