EXERCISES FOR MATHEMATICS 138A

WINTER 2012

Several problems are taken from the following book:

B. O'Neill, *Elementary Differential Geometry* (Second Edition). Academic Press, San Diego, CA, 1997. ISBN: 0–125–26745–2.

I. Classical Differential Geometry of Curves

I.1 : Cross products

1. Verify that the cross product of vectors in $\mathbb{R}^3$ satisfies the *Jacobi identity*:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \; + \; \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) \; + \; \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \quad = \quad \mathbf{0} \; .$$

2. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be orthonormal vectors in $\mathbb{R}^3$ such that $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ (cross product). Compute $\mathbf{v} \times \mathbf{w}$ and $\mathbf{w} \times \mathbf{u}$.

Note. The preceding result has the following consequence: *Suppose that T is a linear transformation on $\mathbb{R}^3$ which takes the standard unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ to the orthonormal vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ respectively. Then we have $T(\mathbf{x} \times \mathbf{y}) = T(\mathbf{x}) \times T(\mathbf{y})$ for all vectors $\mathbf{x}$,$\mathbf{y}$ in $\mathbb{R}^3$.* — The basic idea is merely that if a linear transformation preserves cross products on a basis, then by the Distributive Law of Multiplication it must preserve all cross products.

I.2 : Parametrized curves

O'Neill, § 1.4 (2^{nd} Ed. pp. 21–22): 2, 8

2. Find the unique curve such that $\gamma(0) = (1, 0, 5)$ and $\gamma'(t) = (t^2, t, e^t)$.

8. Sketch the following curves in $\mathbf{R}^2$ and find parametrizations for each:

(a) The set C of all points (x, y) such that $4x^2 + y^2 = 1$.

(b) The set C of all points (x, y) such that $3x + 4y = 1$.

(c) The set C of all points (x, y) such that $y = e^x$.

Additional exercises

1. Find a parametrized curve $\alpha(t)$ which traces out the unit circle about the origin in the coordinate plane and has initial point $\alpha(0) = (0, 1)$.

2. Let $\alpha(t)$ be a parametrized cure which does not pass through the origin. If $\alpha(t_0)$ is the point in the image that is closest to the origin and $\alpha'(t_0) \neq 0$, show that $\alpha(t_0)$ and $\alpha'(t_0)$ are perpendicular.

3. If Γ is the figure 8 curve with parametrization $\gamma(t) = (3\cos t, 2\sin 2t)$, where $0 \leq t \leq 2\pi$, find a nontrivial polynomial $P(x, y)$ such that the image of γ is contained in the set of points where $P(x, y) = 0$. [*Hint:* Recall that $\sin 2t = 2\sin t \cos t$ and $\sin^2 t + \cos^2 t = 1$; the latter implies that $\cos^2 t = \sin^2 t \cos^2 t + \cos^4 t$.]

4. Two objects are moving in the coordinate plane with parametric equations $\mathbf{x}(t) = (t^2 - 2, \frac{1}{2}t^2 - 1)$ and $\mathbf{y}(t) = (t, 5 - t^2)$. Determine when, where, and the angle at which the objects meet.

5.* Prove that a regular smooth curve lies on a straight line if and only if there is a point that lies on all its tangent lines.

I.3 : Arc length and reparametrization

O'NEILL, § 2.2 (2^{nd} Ed. pp. 55–56): 3–5, 10, 11

3. Show that the curve $\alpha(t) = (\cosh t, \sinh t, t)$ has arc length function $s(t) = \sqrt{2}\sinh t$ and find a unit speed reparametrization of α.

4. Consider the curve $\alpha(t) = (2t, t^2, \log t)$ on the interval of all t such that $t > 0$. Show that this curve passes through the points $\mathbf{p} = (2, 1, 0)$ and $\mathbf{q} = (4, 4, \log 2)$, and find its arc length between these points.

5. Suppose that β_1 and β_2 are unit speed reparametrizations of the same curve α. Show that there is a number s_0 such that $\beta_2(s) = \beta_1(s_0 + s)$ for all s. What is the geometric or physical significance of s_0?

10. Let J be some interval in the real line, and let $\alpha, \beta : J \to \mathbf{R}^3$ be curves such that the tangent vectors α' and β' are parallel (same Euclidean coordinates) at each t. Prove that α and β are parallel in the sense that there is a fixed vector $\mathbf{c}$ such that $\beta(t) = \alpha(t) + \mathbf{c}$ for all t.

11. Prove that a straight line is the curve of shortest length in $\mathbf{R}^3$ joining two points as follows: Let $\alpha : [a, b] \to \mathbf{R}^3$ be an arbitrary curve segment from $\mathbf{p}$ to $\mathbf{q}$, and let $\mathbf{u}$ be the unit vector pointing in the same direction as $\mathbf{q} - \mathbf{p}$.

(a) If $\sigma(t)$ is the straight line segment defined by $\sigma(t) = (1 - t)\mathbf{p} + t\mathbf{q}$ for $t \in [0, 1]$, show that its length is equal to $|\mathbf{q} - \mathbf{p}|$.

(b) Using the inequality $|\alpha'| \geq \alpha' \cdot \mathbf{u}$ show that the length of α is greater than or equal to $|\mathbf{q} - \mathbf{p}|$.

(c) Finally, show that if the length of α is $|\mathbf{q} - \mathbf{p}|$, then up to reparametrization α is a straight line segment. [*Hint:* Write $\alpha' = (\alpha' \cdot \mathbf{u}) + \mathbf{N}$ where $\mathbf{N} \cdot \alpha = \mathbf{0}$.]

Additional exercises

0. Find the length of the parametrized plane curve $\mathbf{x}(t) = (\cos^3 t, \sin^3 t)$ from $t = 0$ to $t = 2\pi$.

1. Prove that a necessary and sufficient condition for the plane $\mathbf{N} \cdot \mathbf{x} = 0$ to be parallel to the line $\mathbf{x} = \mathbf{x}_0 + t \cdot \mathbf{u}$ is for $\mathbf{N}$ and $\mathbf{u}$ to be perpendicular.

2.* Suppose that $F(x, y)$ is a function of two variables with continuous partial derivatives such that $F(a, b) = 0$ but $\frac{\partial}{\partial y}F(a, b) \neq 0$, and also suppose that $g(x)$ is a function such that the set

$F(a,b) = 0$ has the parametrization $y = g(x)$ over the interval $[a-h, a+h]$. Prove that the length of this curve is given by the integral

$$\int_{a-h}^{a+h} \frac{|\nabla F(x, g(x)\,)|}{|F_2(x, g(x)\,)|}\, dx$$

where F_2 denotes the partial derivative with respect to the second variable. [*Hint:* Use the implicit differentiation formula for g in terms of the partial derivatives of F.]

3.* (a) Given $a > 0$, consider the set of all continuously differentiable real valued functions f on $[0,1]$ such that $f(0) = 0$ and $f(1) = a > 0$. Define $L(f)$ by the formula $L(f) = \int_0^a |f'(t)|\, dt$. Show that the minimum value of $L(f)$ is a, and if equality holds then f' is everywhere nonnegative. [*Hints:* Since $f' \leq |f'|$ a similar inequality holds for their definite integrals. This inequality of integrals is strict if and only if $f'(t) < |f'(t)|$ for some t, which happens if and only if $f'(t) < 0$ for that choice of t.]

(b) Let $\gamma(t)$ be a regular smooth curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ such that $\gamma(0) = \mathbf{0}$ and $\gamma(1)$ is the first unit vector $\mathbf{e}_1$ with first coordinate equal to 1 and the other coordinate(s) equal to zero. Prove that the length of γ is at least 1, and equality holds if and only if γ is a reparametrization of the straight line segment joining $\gamma(0)$ to $\gamma(1)$. [*Hint:* Write $\gamma = (x, y, z)$ in coordinates, let $\beta = (x, 0, 0)$ and explain why the length of β is less than or equal to the length of γ, with equality if and only if $y = z = 0$. Apply the first part of the problem to show that $x(t)$ defines a reparametrization of the line segment joining the endpoints.

Note. The file `greatcircles.pdf` in the course directory proves the corresponding result for curves of shortest length on the sphere; namely, these shortest curves are given by great circle arcs. As noted in the cited document, the argument uses material from later units in this course, and at several points it is "somewhat advanced." An more elementary proof for the distance minimizing property of great circles can be derived fairly quickly from the first theorem in the online document

`http://math.ucr.edu/~res/math133/polyangles.pdf`

and the standard formula which states that the length of a minor circular arc is equal to the product of the radius of its circle times the measure of its central angle expressed in radians.

4. (*a*) If an object is attached to the edge of a circular wheel and the wheel is rolled along a straight line on a flat surface at a uniform speed, then the curve traced out by the object is a **cycloid** (there is an illustration in the file `cyc-curves.pdf`). If the circle has radius $a > 0$ and its center starts at the point with coordinates $(0, a)$, then the object starts at $(0,0)$ and its parametric equations are given by the classical formula $\mathbf{x}(t) = a \cdot (t - \sin t, 1 - \cos t)$.

Find the length of the cycloid over the parameter values $0 \leq t \leq 2\pi$.

(*b*) In the classical geocentric theory of planetary motion which appears in the *Almagest* of Claudius Ptolemy (*c.* 85–165), there is an assumption that planets travel in curves given by *epicycles.* The simplest examples of these involve circular motion where the center of the circle is moving in a circular path around a second circle (this is similar to the motion of the moon around the earth, which is given by an ellipse while the earth itself is moving around the sun by a larger ellipse; an illustration appears in `cyc-curves.pdf`; in the full theory one also allowed the second circle to move around a third cycle, and so on). A typical example is given by the following formula, in which the first circle has radius $\frac{1}{4}$, the second one is the unit circle about the origin, and the body rotates four times around the small circle as the large circle makes one revolution around its center:

$$\mathbf{x}(t) \quad = \quad (\cos t, \sin t) \ + \ \tfrac{1}{4}\,(\cos 4t, \sin 4t)$$

Find the length of this curve over the parameter values $0 \leq t \leq 2\pi$.

Notes. For both parts of these exercises the standard formulas for $|\sin \frac{1}{2}\theta|$ and $|\cos \frac{1}{2}\theta|$ may be useful.

I.4 : Curvature and torsion

0. Find the curvatures for the graphs of the following functions $f(x)$ using the standard parametrization $(t, f(t), 0)$:
(a) $f(x) = x^3$
(b) $f(x) = \tan x$
(c) $f(x) = e^x$

1. Suppose a curve is given in polar coordinates by $r = r(\theta)$ where $\theta \in [a, b]$.

(i) Show that the arc length is $\int_a^b \sqrt{r^2 + (r')^2}\, d\theta$.

(ii) Show that the curvature is

$$k(\theta) = \frac{2(r')^2 - rr'' + r^2}{[r^2 + (r')^2]^{3/2}} .$$

2. Let α and β be regular parametrized curves such that β is the arc length reparametrization of α. Let t be the parameter for α and s for β. Prove the following:

(a) $dt/ds = 1/|\alpha'|$, $d^2t/ds^2 = -(\alpha' \cdot \alpha'')/|\alpha'|^4$

(b) The curvature is given by

$$k(t) = \frac{\alpha' \times \alpha''}{|\alpha'|^3}$$

(c) The torsion is given by

$$\tau(t) = -\frac{\alpha' \times \alpha'' \cdot \alpha'''}{|\alpha' \times \alpha''|^2}$$

(d) If the plane curve α has coordinate functions x and y, then the signed curvature of α at t is equal to

$$k(t) = \frac{x'y'' - x''y'}{[(x')^2 + (y')^2]^{3/2}}$$

3.* Show that the curvature of a regular parametrized curve α at t_0 is equal to the curvature of the plane curve γ which is the perpendicular projection of α onto the osculating plane of α at t_0.

4. Consider the problem of designing a set of railroad tracks that contains a pair of parallel tracks along with a third going from the first to the second smoothly. Mathematically, the parallel tracks themselves may be viewed as corresponding to the parallel lines $y = 0$ and $y = 1$ in the coordinate plane, and the track going from one to the other may be viewed as a regular smooth curve that is the graph of a twice differentiable function f such that $f(x)$ is zero if $t \leq 0$, $f(x) = 1$ if $t \geq 1$, and on $[0, 1]$ the function f is given by a polynomial $p(x)$. The existence of a second derivative ensures that the slope of the tangent line would be a continuous function of x, and in addition we want to assume that *the curvature is also a continuous function of* x. Find a polynomial $p(x)$ of degree 5 such that all the required conditions are fulfilled. [*Hint:* If we are given a graph curve with parametric equations $(t,\, y(t))$, then the curvature at parameter value t is given by the formula

$$k(t) \quad = \quad \frac{|y''|}{\left(1+(y')^2\right)^{3/2}}$$

and one step in the argument is to use this fact to compute $p''(0)$ and $p''(1)$. In fact, the conditions of the problem uniquely specify the values of p and its first and second derivatives at both 0 and 1. Why does this mean the only values to find are the coefficients of x^3, x^4 and x^5?]

Optional. Graph the function f using calculator or computer graphics.

5. For each of the following curves, compute the curvature as a function of the x-coordinate, find where the curvature takes a maximum value, and explain why the curvature approaches 0 as the x-coordinate approaches the indicated limits.

(a) The hyperbola $y = 1/x$, where $x > 0$ and the limiting values for x are 0 and $+\infty$.

(b) The catenary $y = \cosh x = \frac{1}{2}(e^x + e^{-x})$, where the limiting values for x are $\pm\infty$.

I.5 : Frenet-Serret Formulas

O'NEILL, § 2.3 (2$^{\text{nd}}$ Ed. pp. 64–66): 1, 5

1. Compute the Frenet data $(\kappa, \tau, \mathbf{T}, \mathbf{N}, \mathbf{B})$ for the curve $\gamma(s) = \left(\frac{4}{5}\cos s, 1 - \sin s, -\frac{3}{5}\cos s\right)$. Show this curve is a circle and find its center.

5. If $\mathbf{A}$ is the vector field $\tau\mathbf{T} + \kappa\mathbf{B}$ along the unit speed curve γ, show that the Frenet-Serret formulas become $\mathbf{V}' = \mathbf{A} \times \mathbf{V}$ for $\mathbf{V} = \mathbf{T}$, $\mathbf{N}$ or $\mathbf{B}$.

Additional exercises

1. Let $\mathbf{x}$ be a regular smooth curve with a continuous third derivative, and let $(\mathbf{T},\, \mathbf{N},\, \mathbf{B})$ be its Frenet trihedron. Prove that there is a vector $\mathbf{W}$ (the *Darboux vector*) such that $\mathbf{T}\,' = \mathbf{W} \times \mathbf{T}$, $\mathbf{N}\,' = \mathbf{W} \times \mathbf{N}$, and $\mathbf{B}\,' = \mathbf{W} \times \mathbf{B}$. What is the length of $\mathbf{W}$?

2.* If $\mathbf{x}$ is defined for $t > 0$ by the formula

$$\mathbf{x}(\mathbf{t}) \quad = \quad \left(t,\ \frac{1+t}{t},\ \frac{1-t^2}{t}\right)$$

show that $\mathbf{x}$ is planar.

3. (a) Suppose that D is a diagonal $n \times n$ matrix with diagonal entries $d_1, \cdots, d_n$. Show that $\exp(D)$ is also a diagonal matrix and its entries are $e^{d_1}, \cdots, e^{d_n}$. [*Hint:* If D is diagonal so is every power D^k; what are the latter's diagonal entries?]

(b) The **trace** of a square matrix is defined to be the sum of its diagonal entries. If D is a diagonal matrix with trace t, explain why the determinant of $\exp(D)$ is equal to e^t. [*Note:* This turns out to be true for arbitrary $n \times n$ matrices.]

4. If A and B are square matrices then their **Lie bracket** $[A, B]$ is defined to be $AB - BA$.

(a) Prove that the trace of a Lie bracket matrix is always zero. [*Hint:* Show that the trace is linear function on the space of $n \times n$ matrices and that the traces of AB and BA are equal.

(b) Suppose that A and B are skew-symmetric $n \times n$ matrices (*i.e.*, they are equal to the negatives of their transposes). Show that their Lie bracket is also skew-symmetric. [*Note:* If $n = 3$ then the space of skew-symmetric 3×3 matrices is 3-dimensional and the Lie bracket is essentially the same as the ordinary cross product.]

(c) Let $C_{i,j}$ be the $n \times n$ matrix which has a $+1$ in the (i, j) position, a -1 in the (j, i) position, and zeros elsewhere. Compute the Lie bracket $[C_{1,2}, C_{2,3}]$; in principle, it is only necessary to do this when $n = 3$.

(d) Show that the Lie bracket is anticommutative (algebraically, $[B, A] = -[A, B]$) and also satisfies a version of the Jacobi Identity which holds for cross products:

$$[A, [B, C]] \ + \ [B, [C, A]] \ + \ [C, [A, B]] \quad = \quad 0$$

5. Compute $\exp(N)$ where N is the following 3×3 matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix} .$$

II. Topics from Multivariable Calculus and Geometry

II.1 : Differential forms

There are no assignments related to this section.

II.2 : Smooth mappings

Definition. A subset K of $\mathbb{R}^n$ is said to be *convex* if whenever $\mathbf{x}$ and $\mathbf{y}$ lie in K then the whole line segment defined by the parametrized curve $\mathbf{x} + t\,(\mathbf{y} - \mathbf{x})$ for $t \in [0,1]$ is contained in K.

1. (a) Prove that an open convex set is a connected domain [*Hint:* Imitate the proof for the set of all point whose distance from some point $\mathbf{p}$ is less than some positive number r.].

(b) Describe an example of a connected domain in the plane which is not convex (you do not need to prove that the domain satisfies these conditions).

2. Show by example that an intersection of two connected domains in $\mathbb{R}^2$ is not necessarily a connected domain. [*Hint:* Let U be the annular region defined by the inequalities $1 < x^2 + y^2 < 9$ and let V be the horizontal strip defined by the inequality $|y| < \frac{1}{2}$. Verify that U is arcwise connected using the polar coordinate mapping, which yields a continuous 1–1 mapping from the convex set $(1,3) \times [0, 2\pi)$ onto U. If $U \cap V$ were connected then by a result in the Appendix to Chapter 5 in do Carmo, it would also be arcwise connected. Suppose now that $\mathbf{x}$ is a curve joining the points (± 2.0). By the Intermediate Value Theorem there must be some parameter value t_0 such that the first coordinate of $\mathbf{x}(t_0)$ is equal to zero. Why does this mean that $\mathbf{x}$ cannot lie entirely inside $U \cap V$?]

3. Given an matrix A with real entries, let $|A|$ denote the Euclidean length given by the square root of the standard sum $\sum_{i,j} \, |a_{i,j}|^2$. If P and Q are two matrices with real entries such that the product $P\,Q$ can be defined, prove that $|P\,Q| \leq |P| \cdot |Q|$.

4. Let U be a convex connected domain in $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^m$ be a smooth $\mathcal{C}^1$ function.

(a) Prove that

$$f(\mathbf{y}) - f(\mathbf{x}) \quad = \quad \int_0^1 \; \big(\big[Df\,(\mathbf{x} + t\,(\mathbf{y} - \mathbf{x})) \big] (\mathbf{y} - \mathbf{x}) \,\big) \; dt$$

for all $\mathbf{x}$, $\mathbf{y} \in U$. [*Hint:* Explain why the integrand is the derivative of the function

$$f\,(\mathbf{x} + t\,(\mathbf{y} - \mathbf{x}))$$

using the Chain Rule.]

(b) Suppose that the derivative matrix function Df satisfies $|Df| \leq M$ on U. Prove that

$$|f(\mathbf{y}) - f(\mathbf{x})| \quad \leq \quad M \cdot |\mathbf{y} - \mathbf{x}|$$

for all $\mathbf{x}$, $\mathbf{y} \in U$.

Note. An inequality of this sort is called a *Lipschitz condition.*

5. Find the first order approximation to the function

$$f(x_1, x_2, x_3) \quad = \quad (x_1^2 + x_2^2 + x_3^2, \sin x_2, x_1 x + \cos x_3)$$

at the point $(1, 2, 3)$.

6. Find the first order approximation to the function

$$(u, v) \quad = \quad f(x, y, z) \quad = \quad (xyz^2 - 4y^2, 4y^2, 3xy^2 - y^2 z)$$

at $(1, -2, 3)$.

7. Find a change of variables $u = u(x, y)$, $v = v(x, y)$ which takes the parabola $y = x^2$ to the horizontal axis and the line $y = 3$ to the vertical axis; recall that these axes are defined by $v = 0$ and $u = 0$ respectively.

II.3 : Inverse and Implicit Function Theorems

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^r$ function such that its derivative f' is everywhere positive and the limits of $f(t)$ as $t \to \pm\infty$ are $\pm\infty$ respectively. Prove that f has a $\mathcal{C}^r$ inverse function.

2. Prove that $F(x, y) = (e^x + y,\ x - y)$ defines a 1–1 onto $\mathcal{C}^\infty$ map from $\mathbb{R}^2$ to itself with a $\mathcal{C}^\infty$ inverse.

3. Prove that $F(x, y) = (xe^y + y,\ xe^y - y)$ defines a 1–1 onto $\mathcal{C}^\infty$ map from $\mathbb{R}^2$ to itself with a $\mathcal{C}^\infty$ inverse.

4. (a) Using the change of variables formula, explain briefly why the area of a set in $\mathbb{R}^2$ is the same as the area of its image under a rigid motion of the form $T(\mathbf{x}) = A\,\mathbf{x} + \mathbf{b}$, where A is a rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

(b) More generally, if we are given an arbitrary **affine** transformation as above, where the only condition on A is invertibility, how is the area of a set $\mathcal{F}$ related to the area of its image $T[\mathcal{F}]$?

5. A smooth $\mathcal{C}^r$ mapping f from a connected domain $U \subset \mathbb{R}^2$ into $\mathbb{R}^2$ is said to be *regularly conformal* at $\mathbf{p} = (u_0,\ v_0) \in U$ if the Jacobian of f is positive and for all regular smooth curve pairs $\mathbf{x}$ and $\mathbf{y}$ satisfying $\mathbf{x}(s_0) = \mathbf{y}(s_0) = \mathbf{p}$ the angle between $\mathbf{x}'(s_0)$ and $\mathbf{y}'(s_0)$ is equal to the angle between $[f \circ \mathbf{x}]'(s_0)$ and $[f \circ \mathbf{y}]'(s_0)$.

(a) Prove that the partial derivatives of the coordinate functions satisfy the *Cauchy-Riemann equations*:

$$\frac{\partial f_1}{\partial x_1} \quad = \quad \frac{\partial f_2}{\partial x_2}\,, \qquad\qquad \frac{\partial f_2}{\partial x_1} \quad = \quad -\frac{\partial f_1}{\partial x_2}$$

[*Hint:* If $A = Df(\mathbf{p})$, one needs to show that $\cos \angle(A\,\mathbf{x},\ A\,\mathbf{y}) = \cos \angle(\mathbf{x},\ \mathbf{y})$ for all nonzero vectore $\mathbf{x}$ and $\mathbf{y}$. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ denote the columns of A, and let J denote counterclockwise rotation through $\pi/2$. Why is $\mathbf{a}_2 = c\,J(\mathbf{a}_1)$ for some constant c, and why does the determinant condition imply c is positive? Explain why $A(\mathbf{e}_1 + \mathbf{e}_2) = \mathbf{a}_1 + \mathbf{a}_2$ must be perpendicular to $A(\mathbf{e}_1 - \mathbf{e}_2) = \mathbf{a}_1 - \mathbf{a}_2$, and use this to conclude that $c = 1$.]

(b) There is a modified version of this relation that holds among the partial derivatives if the Jacobian is **negative**. State it and explain why it is true. [*Hint:* Consider what happens if one composes f with the reflection map $S(x, y) = (x, -y)$.]

Note. Functions satisfying the Cauchy-Riemann equations are also known as *complex analytic* functions, and they are the central objects studied in complex variables courses.

6. For what values of (ρ, θ, ϕ) does the spherical coordinate mapping

$$(x, y, z) \quad = \quad (\rho \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi)$$

satisfy the Jacobian condition in the Inverse Function Theorem? Explain why the complement of this set is a line through the origin.

7. The pair of equations $xy + 2z = 3xz$, $xyz + x - y = 1$ has $(1, 1, 1)$ as a solution. Show that this pair of equations can be solved for any two of the unknowns in terms of the third in a vicinity of $(1, 1, 1)$.

8. If $(x, y) = (s^2 - s - 2, 3t)$, use the Implicit Function Theorem to determine values of (s, t) for which this system can be solved locally for s and t.

9. Consider the equation $xz + \sin xy + \cos xz = 1$ near the solution $(0, 1, 1)$. Can it be solved near this point for x? For y? For z?

10. Show that the system

$$e^x + e^{2y} + e^{3u} + e^{4v} \quad = \quad 0\ , \quad e^x + e^y + e^u + e^v \quad = \quad 0$$

has a unique solution for (x, y, z, w) close to $(0, 0, 0, 0)$.

II.4 : Congruence of geometric figures

1. Let F be an isometry of $\mathbb{R}^n$, and let $\mathbf{x}$ and $\mathbf{y}$ be distinct points of $\mathbb{R}^n$ such that $F(\mathbf{x}) = \mathbf{x}$ and $F(\mathbf{y}) = \mathbf{y}$. Suppose that $\mathbf{z}$ is a point on the line joining $\mathbf{x}$ to $\mathbf{y}$ that can be expressed as $\mathbf{z} = t\mathbf{x} + (1 - t)\mathbf{y}$ for some scalar t. Prove that $F(\mathbf{z}) = \mathbf{z}$ also holds. [*Hints:* Use the fact that $F(\mathbf{w}) = A(\mathbf{w}) + \mathbf{b}$ for some linear transformation A along with the identity $\mathbf{b} = t\mathbf{b} + (1 - t)\mathbf{b}$.]

2. Prove that congruent curves have equal lengths.

3. A *similarity transformation* of $\mathbb{R}^n$ is a 1–1 onto mapping of the form $T(\mathbf{x}) = cA\mathbf{x} + \mathbf{b}$, where $c > 0$, A is given by an orthogonal $n \times n$ matrix, and $\mathbf{b}$ is some vector in $\mathbb{R}^n$. If T is a *proper* similarity in the sense that $c \neq 1$ (so that T is not an isometry), then prove that there is a unique vector $\mathbf{v}$ such that $T(\mathbf{v}) = \mathbf{v}$. [*Hint:* This is equivalent to showing that there is a unique solution to the equation $(cA - I)\mathbf{x} = \mathbf{b}$, and a uinque solution of this equation exists if and only if the matrix $cA - I$ is invertible. Why is the latter equivalent to showing that c^{-1} is not an eigenvalue of A, and why do the orthogonality condition on A and $c \neq 1$ imply this fact?]

4.* (*a*) An invertible $n \times n$ matrix A is said to be *conformal* if it preserves angles; *i.e.*, if $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbf{R}^n$ then

$$\cos \angle(\mathbf{x}, \mathbf{y}) \quad = \quad \cos \angle(A\mathbf{x}, A\mathbf{y})$$

where the cosine may defined by the usual inner product formula.

(*b*) Suppose that $A = CB$ where B is orthogonal and $c > 0$. Show that A is conformal.

(*c*) Suppose that A is conformal. Prove that the columns of A are perpendicular. [*Hint:* They define the vectors $A\mathbf{e}_i$ where the $\mathbf{e}_i$ are the standard unit vectors in $\mathbf{R}^n$.]

(*d*) Suppose that L_i is the (positive) length of $A\mathbf{e}_i$. Compute L_i/L_1. [*Hint:* look at the angle between $\mathbf{e}_1 + \mathbf{e}_i$ and $\mathbf{e}_1$ and the angle between the images of these vectors under A.]

(*e*) Why do the preceding two parts of the problem imply that if A is conformal then $A = cB$ where B is orthogonal and $c > 0$?

(*f*) Let f be the map from $\mathbb{R}^2$ to itself defined by $f(u, v) = (u^2 - v^2, 2uv)$. Prove that $Df(u, v)$ is conformal for all $(u, v) \neq (0, 0)$. Do the same for $g(u, v) = (u^3 - 3uv^2, 3u^2v - v^3)$. For both of these exercises, it is helpful to use the Cauchy-Riemann equations from a previous exercise.

(*g*) Prove that a similarity transformation is conformal.

III. Surfaces in 3-Dimensional Space

III.1 : Mathematical descriptions of surfaces

O'NEILL, § 4.1 (2^{nd} Ed. pp. 132–133): 1, 4*bc*, 5, 9

4(b, c)**.** For which of the following is $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ a 1–1 regular parametrization (so that $\sigma(u, v) = \sigma(u', v')$ if and only if $(u, v) = (u', v')$)?

(b) $\sigma(u, v) = (u^2, u^3, v)$.

(c) $\sigma(u, v) = (u, v^2, v^3 + v)$.

5. (a) Prove that the set of all points (x, y, z) satisfying $(x^2 + y^2)^2 + 3z^2 = 1$ is a regular geometric surface.

(b) For which values of c is the set of all points satisfying $z(z - 2) + xy = c$ a regular surface?

9. Let $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ be the mapping $\sigma(u, v) = (u + v, u - v, uv)$. Show that σ is a regular surface parametrization that is 1–1 and that the image of σ is the entire surface defined by the equation $z = \frac{1}{4}(x^2 - y^2)$.

Additional exercises

1. Write down equations defining the surfaces given by the following geometric conditions:

(a) The set of points that are equidistant from the point $(0, 0, 4)$ and the xy-plane.

(b) The set of points that are equidistant from the point $(0, 2, 0)$ and the plane defined by the equation $y = -2$.

(c) The set of points that are equidistant from the points $(0, 0, 0)$ and $(1, 0, 0)$.

(d) The set of points for which the sum of the distances to $(\pm 1, 0, 0)$ is equal to 5.

2. Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be linearly independent vectors in $\mathbb{R}^3$. Prove that there is a unique sphere containing these three points and $\mathbf{0}$; *i.e.*, show that the system of equations

$$|\mathbf{x} - \mathbf{a}|^2 \quad = \quad |\mathbf{x} - \mathbf{b}|^2 \quad = \quad |\mathbf{x} - \mathbf{c}|^2 \quad = \quad |\mathbf{x}|^2$$

has a unique solution $\mathbf{x}$.

3. If $\mathbf{X}(u, v) = (u^2 - v^2, u - v, u^3 + 3v) = (x, y, z)$, find a nontrivial polynomial $P(x, y, z)$ such that the image of $\mathbf{X}$ is contained in the set of points where $P(x, y, z) = 0$.

4. If $\mathbf{X}(u, v) = (u, u^2 + v, v^2)$, find a nontrivial polynomial $P(x, y, z)$ such that the image of $\mathbf{X}$ is contained in the set of points where $P(x, y, z) = 0$.

III.2 : Parametrizations of surfaces

0. Describe parametric equations for the surface obtained by rotating the curve $y = e^{-x}$ around the x-axis.

1. Let $f(x, y, z) = (x + y + z - 1)^2$.

(i) What are the critical points and values?

(ii) For which c is the level set for c a regular surface?

(iii) Same questions for xyz^2.

2. Let Σ be a geometric regular smooth surface, let U be a connected domain in $\mathbb{R}^3$ containing Σ, and let $\mathbf{g} : U \to \mathbb{R}^3$ be a smooth 1–1 onto map such that the Jacobian of $\mathbf{g}$ is nowhere zero (hence it has a global inverse), its image is a connected domain, and more generally the image of any connected subdomain of U is also a connected domain. Prove that $\mathbf{g}(\Sigma)$ is also a geometric regular smooth surface.

3. Suppose we are given a positive valued function $f(\theta)$ with continuous first two derivatives, and suppose that we consider the set S defined by the cylindrical coordinate equation $r = f(\theta)$, so that it has a parametrization of the form $\mathbf{X}(r, \theta) = \big(f(\theta)\cos\theta, f(\theta)\sin\theta, z\big)$. Show that this is a regular parametrization and find a normal vector to the tangent plane for $\mathbf{X}(r, \theta)$.

4. Suppose now that we are given a positive valued function $f(\theta, \phi)$ with continuous partial derivatives, and consider the set defined by the spherical coordinate equation $\rho = f(\theta, \phi)$, so that a parametrization is given by

$$\mathbf{X}(\theta, \phi) \quad = \quad \big(f(\theta,\phi)\cos\theta\sin\phi, f(\theta,\phi)\sin\theta\sin\phi, f(\theta,\phi)\cos\phi\big) \ .$$

Suppose that f is defined for (θ, ϕ) close to $(0, \frac{1}{2}\pi)$. Find the normal direction for the tangent plane to the surface when $(\theta, \phi) = (0, \frac{1}{2}\pi)$, and prove that the normal direction at $(0, \frac{1}{2}\pi)$ is given by $\pm\,(1, 0, 0)$ if and only if $\nabla f(0, \frac{1}{2}\pi) = \mathbf{0}$.

5. Consider the mapping

$$\sigma(u, v) \quad = \quad \left(\frac{u^2}{1+v^3}, \frac{u^2 v}{1+v^3}, \frac{u^3}{1+v^3}\right)$$

and take the domain to be the set of all points in the open first quadrant (so that $u, v > 0$). Prove that σ is a 1–1 regular parametrization and its image is the surface in the open first octant (points whose three coordinates are all strictly positive) defined by the equation $x^3 + y^3 = z^2$. Why does this imply that the latter equation has infinitely many solutions for which x, y, z are all positive integers? [*Note:* A basic number-theoretic result of P. de Fermat (1601–1655) states that there are no solutions to $x^4 + y^4 = z^2$ for which x, y, z are all positive integers.]

III.3 : Tangent planes

O'NEILL, § 4.3 (2^{nd} Ed. pp. pp. 150–153): 6*bc*, 10

10. In each of the cases below find an equation of the form $ax + by + cz = d$ (with $(a, b, c) \neq (0, 0, 0)$ for the tangent plane:

(a) The sphere defined by $x^2 + y^2 + (z-1)^2 = 1$ at the point $(0, 0, 0)$.

(b) The ellipsoid defined by

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{64} \quad = \quad 1$$

at the point $(1, -2, 3)$.

(c) The helicoid parametrized by $\sigma(u,v) = (u\cos v, u\sin v, 2v)$ at the point $\sigma(2,\pi/4)$.

Additional exercises

0. Find equations defining the tangent planes to the given surfaces at the indicated points:

(a) The parametrized surface $\mathbf{X}(u,v) = (u, v, \sqrt{uv})$ at $(1,1,1)$

(b) The parametrized surface $\mathbf{X}(u,v) = (2u\cos v, 3u\sin v, u^2)$ at $(0,6,4)$. [You need to find (u,v) in this example.]

(c) The parametrized surface $\mathbf{X}(u,v) = (2u\cosh v, 3u\sinh v, \frac{1}{2}u^2)$ at $(-4,0,2)$. [Same note as in the previous exercise.]

1. Show that the tangent plane is the same at all points along a ruling of a cylinder.

Definition. A surface S is said to be *globally convex* at a point $\mathbf{p}$ if all points of S lie on one of the half planes determined by this tangent plane at $\mathbf{p}$ (*i.e.*, if the equation of the tangent plane is $\mathbf{a}\cdot\mathbf{x} = b$, then the points of the surface are completely contained in the set determined by the inequality $\mathbf{a}\cdot\mathbf{x} \le b$ or the reverse inequality $\mathbf{a}\cdot\mathbf{x} \ge b$). A surface is said to be strictly globally convex if in addition for each point $\mathbf{p}$ the intersection of S with the tangent plane consists only of the point $\mathbf{p}$.

The surface S is said to be *locally convex* or strictly locally convex at $\mathbf{p}$ if there is an open disk D containing $\mathbf{p}$ such that $S \cap D$ is globally convex or strictly globally convex.

2.* Let $\mathbf{X}$ be a parametrized surface defined on a connected domain U, and let $(a,b) \in U$. Define a level function $L(u,v)$ by $L(u,v) = [\mathbf{X}(u,v),\ \mathbf{X}_u(a,b),\ \mathbf{X}_v(a,b)]$ (the vector triple product).

(a) Explain why the surface is locally convex at $\mathbf{p} = \mathbf{X}(a,b)$ if and only if L has a relative maximum or minimum at (a,b) and why the surface is strictly locally convex there if and only if L has a strict relative maximum or minimum.

(b) Why does the gradient of L vanish at (a,b)?

(c) If $H(a,b)$ is the determinant

$$\begin{vmatrix} [\mathbf{X}_{u,u}(a,b),\ \mathbf{X}_u(a,b),\ \mathbf{X}_v(a,b)] & [\mathbf{X}_{u,v}(a,b),\ \mathbf{X}_u(a,b),\ \mathbf{X}_v(a,b)] \\ [\mathbf{X}_{v,u}(a,b),\ \mathbf{X}_u(a,b),\ \mathbf{X}_v(a,b)] & [\mathbf{X}_{v,v}(a,b),\ \mathbf{X}_u(a,b),\ \mathbf{X}_v(a,b)] \end{vmatrix}$$

explain why a surface is **NOT** locally convex at $\mathbf{p}$ if $H(a,b) < 0$. [*Hint:* Why does L have a saddle point at (a,b)?]

(d) In the notation of the preceding part of the problem, show that the surface is strictly locally convex at $\mathbf{p}$ if $H(a,b) > 0$. [*Hint:* Why does L have a strict local maximum or minimum?]

(e) If $\mathbf{X}$ is a graph parametrization of the form $\mathbf{X}(u,v) = (u, v, f(u,v))$, prove that $H(a,b)$ is a 2×2 determinant of a matrix whose entries are the corresponding second partial derivatives of f at (a,b).

(f) Apply the preceding to show that if $p \ge 2$ then the graph of the function

$$z = (1 - |x|^p - |y|^p)^{1/p}$$

is strictly locally convex at all (x,y) such that $|x|^p + |y|^p < 1$. In particular, the case $p = 2$ merely states that the usual sphere is strictly locally convex at each point (in fact, all these surfaces are *globally* strictly convex, but we shall not attempt to prove this). [*Hint:* If $r > 1$, explain why the derivative of $|x|^r$ is equal to $r\,|x|^{r-1}$. There are three cases, depending upon whether x is positive, negative or zero.]

NOTE. By interchanging the roles of the three coordinates in the preceding result one can in fact show that the sets defined by the equations $|x|^p + |y|^p + |z|^p = 1$ are all regular smooth surfaces and are strictly locally convex at all points.

Further study. Graph the intersection of this surface with the xz-plane for $p = 3$ and 4 using calculator or computer graphics. Try this also for larger values of p and describe the limit of these surfaces as $p \to \infty$.

3.* Let S be the cylindrical surface with parametric equation(s) $\mathbf{X}(u, v) = (u \cos u, u \sin u, v)$ for $u \in (\pi/2,\, 9\pi/2)$ and $v \in (-1, 1)$. This is a cylinder generated by the Archimedean spiral curve in the plane given in polar coordinates by $r = \theta$. Show that S is locally convex at each point but not globally convex at some point in S (for example, at $(2\pi, 0, 0)$). [*Hints:* Use the results of the preceding exercise to show that the surface is locally convex, and draw a sketch to show that there are points of this curve which lie on both sides of the tangent line to the curve at $(2\pi, 0, 0)$. Can you use this to find two points on the curve which lie on opposite sides of the tangent line?]

NOTE. One can modify the example in this exercise to get a surface that is strictly locally convex but not globally convex at $(2\pi, 0, 0)$) by taking $\sin v$ rather than v to be the third coordinate.

4.* For each of the following quadric surfaces, use the conclusion of Exercise 2 to determine the sets of points $\mathbf{p}$ where the surface is locally convex and where it is strictly locally convex.

(a) The hyperboloid of two sheets defined by the equation $z^2 - x^2 - y^2 = 1$, where the two pieces are parametrized by $\mathbf{X}(u, v) = (\sinh v\, \cos u, \sinh v\, \sin u, \pm \cosh v)$.

(b) The hyperboloid of one sheet defined by the equation $x^2 + y^2 - z^2 = 1$, parametrized by $\mathbf{X}(u, v) = (\cosh v\, \cos u, \cosh v\, \sin u, \sinh v)$.

(c) The elliptic paraboloid defined by the equation $z = x^2 + y^2$.

(d) The hyperbolic paraboloid defined by the equation $z = y^2 - x^2$.

4. Determine the tangent planes to the surface $x^2 + y^2 - z^2 = 1$ at all points $(x, y, 0)$ and show they are all parallel to the z-axis.

5. Let f be a smooth function. Show that the tangent planes to the surface $z = xf(y/x)$, where $x \neq 0$, all pass through the origin.

6.* Show that if all the normals to a connected surface S pass through some point, then the surface is part of a sphere. [Here is a version of the problem not involving connectedness: Under the given conditions, prove that at each point $\mathbf{p}$ of the surface S, then some neighborhood of $\mathbf{p}$ in S is contained in a sphere. — It is helpful to translate the surface so that the common point is the origin.]

7. Show that the tangent planes of the common points for the spheres defined by $|\mathbf{x}|^2 = 1$ and $|\mathbf{x} - \mathbf{a}|^2 = 1$ are perpendicular if and only if $|\mathbf{a}|^2 = 2$. How does this generalize if the radius of one sphere is r and the radius of the other sphere is s?

8. Let d be an odd integer, let p, q, r be nonzero real numbers, and let $f(x, y, z) = px^d + qy^d + rz^d$.

(a) Show that $f(a\mathbf{v}) = a^d f(\mathbf{v})$ for all $a > 0$ and $\mathbf{v}$ in $\mathbb{R}^3$, and also show that $\nabla f(x, y, z) = \mathbf{0}$ if and only if $x = y = z = 0$. Why does this imply that for all $c > 0$ the set $S(c)$ of all (x, y, z) such that $f(x, y, z) = c$ is a geometric surface?

(*b*) Suppose that $\mathbf{w} \in S(1)$, so that if $a > 0$ we have $a\mathbf{w} \in S(a^d)$. Let $P(1)$ and $P(a)$ denote the tangent planes to $S(1)$ and $S(a^d)$ at $\mathbf{w}$ and $a\mathbf{w}$ respectively. Prove that either $P(1)$ and $P(a)$ are parallel or else the 1-dimensional vector subspace spanned by $\mathbf{w}$ lies in both tangent planes, and that the second option arises only if $\mathbf{w}$ and $\nabla f(\mathbf{w})$ are perpendicular. [*Hints:* Compare the normal directions for $P(1)$ and $P(a)$, and explain why they are the same. Why does this mean that the two tangent planes are either parallel or equal? Also, recall that if a line and a plane have a point in common, then the line is contained in the plane if and only if a direction vector for the line and a normal direction vector for the plane are perpendicular.]

III.4 : The First Fundamental Form

1. Show that the first fundamental form on the surface of revolution

$$\mathbf{X}(u,v) \quad = \quad (\,f(u)\cos v,\ f(u)\sin v,\ g(v)\,)$$

is given by $(f')^2\,du\,du + (\,f^2 + (g')^2\,)\,dv\,dv$.

2. If the first fundamental form on a parametrized patch has the form $du\,du + f(u,v)\,dv\,dv$, prove that the v-parameter curves cut off equal segments on all u-parameter curves (the former are the curves where the v coordinate is held constant, and the latter are the curves for which the u coordinate is held constant).

3. Compute the first fundamental forms of the following parametrized surfaces at points where they are regular.

(*i*) The ellipsoid $(a\sin u\cos v, b\sin u\sin v, c\cos u)$.

(*ii*) The elliptic paraboloid $(au\cos v, bu\sin v, u^2)$.

(*iii*) The hyperbolic paraboloid $(au\cosh v, bu\sinh v, u^2)$.

(*iv*) The two sheeted hyperboloid $(a\sinh u\cos v, b\sinh u\sin v, c\cosh u)$.

(*v*) The upper half of the cone $(z\cos v, z\sin v, z)$; in other words, the set of all points on this cone for which $z > 0$.

4. Show that a surface of revolution about the x-axis can be parametrized so that $E = E(v)$, $F = 0$, $G = 1$.

III.5 : Surface area

1. Find the area of the corkscrew surface with parametrization $\mathbf{X}(r,\theta) = (r\,\cos\theta,\ r\,\sin\theta, \theta)$ for $1 \le r \le 2$ and $0 \le \theta \le 2\pi$.

2. Find the area of the parametrized Möbius strip

$$\mathbf{X}(u,v) \quad = \quad (\cos u,\,\sin u, 0) \ + \ v\cdot\big(\cos u\,\cos(u/2),\,\sin u\,\cos(u/2),\,\sin(u/2)\,\big)$$

where $u \in (0, 2\pi)$ and $v \in (-h, h)$ with $0 < h < \frac{1}{2}$. You may view the area as being given by an integral over $[0, 2\pi] \times [-h, h]$.

III.6 : Curves as surface intersections

1. The twisted cubic with parametric equations (t, t^2, t^3) is the intersection of the cylindrical surfaces defined by the equations $z - x^3 = 0$ and $y - x^2 = 0$. What is the angle between the gradients of these functions at the point (x, x^2, x^3)?

2. Show that the parametrized curve $\mathbf{x}(\theta) = (1 + \cos\theta, \sin\theta, 2\sin(\theta/2))$ is regular and lies on the sphere of radius 2 about the origin and the cylinder $(x-1)^2 + y^2 = 1$. Also show that the normal vectors to the two surfaces are linearly independent at the points of intersection if $y \neq 0$.

3. Let f and g be two functions with continuous derivatives defined on the open unit disk $u^2 + v^2 < 1$, and suppose there is a point (a, b) in this open disk where $f(a, b) = c = g(a, b)$, so that the graphs of the surfaces intersect at (a, b, c). Prove that the intersection is transverse if and only if $\nabla f(a, b) \times \nabla g(a, b) \neq \mathbf{0}$.

4.* Suppose that $\gamma(s)$ is a regular smooth curve with nonzero curvature everywhere, and suppose that the parametrization is in terms of arc length plus a constant. Let $\mathbf{T}(s)$, $\mathbf{N}(s)$, and $\mathbf{B}(s)$ denote the Frenet trihedron for γ. Explain why $\mathbf{X}(s, u) = \gamma(s) + u\mathbf{N}(s)$ and $\mathbf{Y}(s, v) = \gamma(s) + v\mathbf{B}(s)$ define ruled surfaces such that near some arbitrary point $\gamma(s_0)$ the intersection of their images is equal to the image γ. [As usual, "near some point" means that there is a small open set W containing the point such that the statement is true for points in W.]

III.7 : Map projections

1. Using the definition of the stereographic equations and the formula for the associated map $F : S^2 - \{(0, 0, 1)\} \to \mathbb{R}^2$ sending (x, y, z) to $(u(x, y), v(x, y))$, express x, y, z as functions of u and v. [*Hint:* The geometry of the problem implies that

$$(x, y, z) \quad = \quad (1 - s) \cdot (0, 0, 1) \ + \ s \cdot (u, v, -1)$$

where s must be chosen so that the vector on the right hand side has length (squared) equal to 1. Solve for s as a function of u and v.]

2. If we look at the stereographic map centered at the North Pole which takes the latter to the origin, it seems obvious that meridians in the sphere correspond to lines through the origin in the uv-plane and latitudinal circles in the sphere correspond to circles in the uv-plane centered at the origin. Prove that these observations are mathematically correct. [*Hint:* The meridian curve for longitude θ is given by the equation $y = x \tan\theta$ and the latitudinale curves are the intersection of the sphere with the planes $z = c$ where c ranges strictly between 1 and -1. The radius of the circle in the uv-plane will be a rational function of c.]

Note. The file `invstero.pdf` establishes another important property of stereographic projections; namely, they preserve the angles a which regular smooth curves intersect (*i.e.*, they are **conformal**).

IV. Oriented Surfaces

IV.1 : Normal directions and Gauss maps

1. What are the images of the Gauss maps for the following surfaces? Take the unit normals defined by positive multiples of the corresponding functions' gradients.

(i) The hyperbolic cylinder defined by the equation $xy = 1$.

(ii) The paraboloid of revolution defined by the equation $z = x^2 + y^2$.

(iii) The hyperbolic paraboloid (saddle surface) defined by the equation $z = x^2 - y^2$.

(iv) The Möbius strip defined by the parametrization

$$\mathbf{X}(u, v) \quad = \quad (\cos u,\, \sin u, 0) \;+\; v \cdot \Big(\cos u \cos(u/2),\, \sin u \cos(u/2),\, \sin(u/2)\Big)$$

where $u \in \mathbb{R}$ and $v \in (-1, 1)$.

2. Prove that the image of the Gauss map for the ellipsoid

$$x^2 \;+\; \frac{y^2}{a^2} \;+\; \frac{z^2}{b^2} \quad = \quad 1$$

(where $a, b > 0$) is the entire sphere. [*Hint:* A normal vector field over the surface is given by $(2x, 2y/a^2, 2z/b^2)$. Using this, reduce the problem to showing that if $u^2 + v^2 + w^2 = 1$, then one can find x, y, z, k such that $k > 0$, (x, y, z) lie on the ellipsoid, and $(u, v, w) = k \cdot (2x, 2y/a^2, 2z/b^2)$. Show these equations have a solution; try expressing x, y, z in terms of u, v, w and solving for k.]

3. Let $(\Sigma, \mathbf{N})$ be an oriented surface in $\mathbb{R}^3$. Prove that if the image of the Gauss map for $(\Sigma, \mathbf{N})$ is all of the unit sphere S^2, then every plane in $\mathbb{R}^3$ is parallel to a tangent plane for Σ at one or more of its points. Is the converse true? Prove it or give a counterexample.

4. Suppose we are given a regular smooth curve $\gamma(t)$ with coordinate functions $x(t)$ and $y(t)$ in the open first quadrant of the coordinate plane (in other words, both coordiantes are positive). Let S be the surface of revolution obtained by rotating this curve about the y-axis, so that it has a parametrization fo the form

$$\big(\, x(t)\cos\theta, y(t), x(t)\sin\theta \,\big) \;.$$

(a) Suppose we know that $(A, B, 0)$ is a unit normal vector to the curve for some parameter value t_0. Prove that the entire circle $(a\cos\theta, b, a\sin\theta)$ lies in the image of the Gauss map.

(b) Suppose now that $\gamma(t)$ is the standard parametrization of the circle $(x-2)^2 + (y-2)^2 = 1$, so that $x = 2 + \cos t$ and $y = 2 + \sin t$. Show that every unit vector in the xy-plane is the normal vector to this curve for some parameter value t_0, and using this and the first part of the problem show that the Gauss map for the torus (doughnut shaped surface), which we take as given by rotating the curve about the y-axis, is onto the unit 2-sphere.

IV.2 : The Second Fundamental Form

O'NEILL, § 5.1 (2^{nd} Ed. pp. 200–201): $3bd$

3(bd). For each of the surfaces below, find the rank of the shape operator $\mathbf{S}$ at the point $(0,0,0)$:

(b) The surface defined by $z = 2x^2 + y^2$.

(d) The surface defined by $z = xy^2$.

Additional exercise

1. Suppose that Σ is an oriented surface whose Second Fundamental Form is identically zero. Show that (locally) Σ is contained in some plane.

IV.3 : Quadratic forms and adjoint transformations

1. Let A be a symmetric 2×2 matrix.

(i) Show that A has two positive eigenvalues if and only if $a_{1,1}$ and $\det A$ are both positive.

(ii) Show that A has one positive and one negative eigenvalue if and only if $\det A$ is negative.

(iii) Show that A has one zero eigenvalue and one positive eigenvalue if and only if $\det A = 0$ and the trace of A is positive.

(iv) How do the criteria in (i) and (iii) change if *positive* is replaced by *negative* in the condition on eigenvalues?

2. Let A be a symmetric 3×3 matrix, and let B be the 2×2 matrix obtained by deleting the third row and column of A. As indicated in the notes, it follows that A has an orthonormal basis of eigenvectors. Suppose that all of the eigenvectors are positive.

(i) Explain why the determinant of A is positive.

(ii) Explain why B also has positive eigenvalues and hence a positive determinant. [*Hint:* Look at the quadratic form in two variables defined by the symmetric matrix B. Why is it positive except at $(0,0)$, where the value ix 0? What does this mean for the eigenvalues of B?]

Note. A basic result in linear algebra called the *Principal Minors Criterion* gives a converse to the preceding results; in the 3×3 case, it states that if A is a symmetric matrix such that $\det A > 0$, $\det B > 0$ and $a_{1,1} = b_{1,1} > 0$, then all the eigenvalues for A are positive. A proof of this fact is essentially given in the following online document:

`http://math.ucr.edu/ res/math132/linalgnotes.pdf`

The first step is to prove a version of Rayleigh's Principle for 3×3 matrices: The minimum and maximum values of the quadratic form determined by A for vectors of unit length are given by the maximum and minimum eigenvalues. Thus the eigenvalues of the matrix are all positive if and only if the value of the quadratic form is positive for all nonzero choices of variables; when this happens we say that the symmetric matrix A is *positive definite*. One can then combine this equivalence with the arguments on pages 84 and 89–90 in the displayed reference to obtain the conclusion described above and its generalization to symmetric $n \times n$ matrices for all values of n.

IV.4 : Normal, Gaussian and mean curvature

O'NEILL, § 5.3 (2$^{\text{nd}}$ Ed. pp. 213–216): 3

3. Prove the following:

(a) The average value of the normal curvature in **any** two orthogonal directions at a point $\mathbf{p}$ is the mean curvature $H(\mathbf{p})$.

(b) The mean curvature $H(\mathbf{p})$ is given by the integral

$$\frac{1}{2\pi} \int_0^{2\pi} k(\theta)\, d\theta$$

where $k(\theta)$ is the normal curvature for the planar section curve given by the tangent vector $\cos\theta\,\mathbf{u}_1 + \sin\theta\,\mathbf{u}_2$ and $\mathbf{u}_1$, $\mathbf{u}_2$ are some orthonormal basis for the space of tangent vectors at $\mathbf{p}$.

Additional exercises

1. Complete the computations of the Gaussian and mean curvatures for the hyperboloids of one and two sheets, the ellipsoid, the hyperbolic and elliptic paraboloids, and the Möbius strip. (NOTE: Since the Möbius strip is not orientable it is only meaningful to discuss the absolute value of the mean curvature, which can be computed locally using any given local orientation),

2. (a) Suppose that $\mathbf{p}$ is a point on the (oriented) surface Σ at a maximum distance from the origin. Prove that the Gaussian curvature at $\mathbf{p}$ is positive.

(b) Suppose that $\mathbf{p}$ is a point on Σ such that the function on Σ whose x-coordinate assumes a maximum value. Prove that the Gaussian curvature at $\mathbf{p}$ is nonnegative, and give an example to show that it is not necessarily positive. [*Hint:* If M is the maximum value, then all points of the surface lie on one closed side of the plane $x = M$. Why must this be the tangent plane to the surface at $\mathbf{p}$?]

3. Suppose that $\mathbf{p}$ is a common point on two surfaces Σ_1 and Σ_2 such that the normals of the two surfaces at $\mathbf{p}$ are linearly independent. Let C be the curve through $\mathbf{p}$ given by the intersection of Σ_1 and Σ_2. Prove that the curvature κ at $\mathbf{p}$ for this curve satisfies

$$\kappa^2 \sin^2\alpha \quad = \quad \kappa_1^2 \; + \; \kappa_2^2 \; - \; 2\,\kappa_1\,\kappa_2 \,\cos\alpha$$

where κ_1 and κ_2 are the normal curvatures of the surfaces in the direction of C at $\mathbf{p}$ and α is the angle between the normals to the surfaces at $\mathbf{p}$.

4. The *Third Fundamental Form* of an oriented surface is defined by

$$\mathbf{III}(\mathbf{x}, \mathbf{y}\,) \quad = \quad \langle D\,\mathbf{N}(\mathbf{p})](\mathbf{x}),\, D\,\mathbf{N}(\mathbf{p})](\mathbf{y}),\, \rangle \ .$$

Prove that $\mathbf{III} - 2\,H\,\mathbf{II} + K\,\mathbf{I} = 0$ where H and K are the mean and Gaussian curvatures. [*Hint:* If A is a diagonalizable matrix explain why $A^2 - \text{trace}(A)\,A + (\det A)\,I = 0$ and use the fact that if T is a self adjoint linear transformation then $\langle T(\mathbf{x}),\, T(\mathbf{y})\,\rangle = \langle T^2(\mathbf{x}),\, \mathbf{y}\,\rangle$.]

5. Assume that a surface Σ has the property that the principal curvatures $\kappa_\pm$ satisfy $|\kappa_\pm| \le 1$. Does it also follow that curvature of a curve on Σ also satisfies $|\kappa| \le 1$?

6. Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.

7. Show that if the mean curvature H is identically zero on Σ and the latter has no planar points, then the Gauss map from Σ to S^2 has the following property:

$$\langle DN_p(w_1),\, DN_p(w_2)\rangle \quad = \quad -k(p)\langle w_1, w_2\rangle$$

for all tangent vectors $w_i \in T_p(\Sigma)$. Show that the above condition implies that the angle of two intersecting curves on S^2 and the angle of their spherical images are equal up to sign.

8. Consider the following parametrized surface, known as *Enneper's surface*:

$$\mathbf{X}(u,v) \quad = \quad \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 = v^2\right)$$

(a) Show that the coefficients of the First Fundamental Form are $E = G = (1+u^2+v^2)^2$ and $F = 0$.

(b) Show that the coefficients of the Second Fundamental Form are $e = -g = 2$ and $f = 0$.

(c) Show that the principal curvatures are $\pm 2/E = \pm 2/G$.

9. Suppose that Σ is a regular surface in $\mathbb{R}^3$ and $F : \mathbb{R}^3 \to \mathbb{R}^3$ is the similarity map sending each $\mathbf{x} \in \mathbb{R}^3$ to $c\mathbf{x}$ where c is a fixed positive real number. Let $\Sigma' = F(\Sigma)$. How are the mean and Gaussian curvatures of Σ and Σ' related?

10. Suppose that Σ is a surface with Gaussian curvature $K > 0$ everywhere. Let Γ be a regular smooth curve in Σ. Prove that the unsigned curvature of Γ is everywhere positive (remember that the unsigned curvature is always nonnegative).

11. Let $\mathbf{a}(t)$ be a regular smooth plane curve (so its third coordinate vanishes). Then the cone surface on $\mathbf{a}$ with vertex equal to the third unit vector $\mathbf{e}_3 = (0,0,1)$ is given by the parametrization $(1-v)\mathbf{a}(u) + v\mathbf{e}_3$ or equivalently in the ruled surface form $\mathbf{a}(u) + v(\mathbf{e}_3 - \mathbf{a}(u)\,)$.

(a) Show that this is a regular parametrization for $v < 1$.

(b) Compute the local First and Second Fundamental Forms in terms of the given parametrization. The answer should be expressed in terms of the variables u and v, the vector valued functions $\mathbf{a}(u)$ and $\mathbf{b}(u)$, and the first and second derivatives of these functions.

(c) Compute the Gaussian curvature for the surface described above in terms of the same quantities as in (a).

(d) Compute the mean curvature if $\mathbf{a}$ is the circle $(c + r\cos t, r\sin t)$, where $r > 0$ and c is a real number, and also for the piece of the parabola $y = x^2 - 1$ for $|x| \le 1$.

(e) Set up the integrals for computing the areas of the portions of the surfaces in (c) lying between the planes $z = 1$ and $z = 0$. You do not need to evaluate the integrals.

12. Let A be a symmetric 2×2 matrix. An *asymptotic vector* for A is a nonzero vector $\mathbf{v}$ such that $\langle A\mathbf{v}, \mathbf{v}\rangle = 0$.

(a) Suppose that A is an invertible matrix such that $\det A > 0$. Explain why A cannot have any asymptotic vectors. [*Hint:* Use Rayleigh's principle. What can one say about the eigenvalues of A in this case using the positivity of $\det A$?]

(b) Suppose now that A is invertible and $\det A < 0$. Prove that A has two linearly independent asymptotic vectors such that every asymptotic vector is a multiple of one of these vectors. [*Hint:*

Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be orthonormal eigenvectors for A with corresponding eigenvalues $\lambda_1 \geq \lambda_2$. What does the determinant condition imply about the signs of these vectors? Given a nonzero vector $\mathbf{v} = x\mathbf{u}_1 + y\mathbf{u}_2$, find a necessary and sufficient condition on x and y for $\mathbf{v}$ to be an asymptotic vector in terms of λ_1 and λ_2. You should get conditions of the form $y = \pm\, c\, x$ for some nonzero constant c. What is it?]

(c) Suppose that A as in the preceding part of the problem and $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent asymptotic vectors for A. Express the absolute value of the cosine of the angle between these vectors in terms of the eigenvalues. [*Note:* Since $\mathbf{v}_1$ and $-\mathbf{v}_2$ are also linearly independent asymptotic vectors, only the absolute value of the cosine is independent of the choices for $\mathbf{v}_1$ and $\mathbf{v}_2$.]

(d) Continuing with the setting of the previous two parts of the problem, give a necessary and sufficient condition on A for asymptotic vectors to be perpendicular to each other.

Motivation for Problem 12. The relevance to differential geometry arises from the notion of *asymptotic curves* in a surface with negative Gaussian curvature. These curves, which play an important role in the study of negatively curved surfaces, are regular smooth curves γ in an oriented surface such that for each parameter value t we have

$$0 \quad = \quad \langle \mathbf{S}_{\gamma(t)}\, \gamma'(t),\, \gamma'(t)\rangle$$

where (as usual) $\mathbf{S}$ denotes the shape operator for the oriented surface. Further information on such curves (and many other important types of curves in a surface) is given in O'Neill.

IV.5 : Special classes of surfaces

O'Neill, § 5.4 (2nd Ed. pp. 222–227): 7, 16ab, 17

7. Find the curvature of the monkey saddle defined by $z = x^3 - 3x^2y$ and express it in terms of $r = \sqrt{x^2+y^2}$.

16(ab)**.** (*Loxodromes*) For $a \neq 0$ let $f_a : (-\pi/2, \pi/2) \to \mathbb{R}$ be the unique function such that $f' - a(t) = at\cos t$ and $f_a(0) = 0$. If $\mathbf{X}$ is the geographic parametrization of the sphere by spherical coordinates, the curve $\lambda_a(t) = \mathbf{x}(f_a(t), t)$ is called a **loxodrome**.

(a) Prove that λ_a' always makes a constant angle with the due north-south vector field $\mathbf{X}_\phi$. Thus λ_a represents a trip with constant (idealized) compass bearing.

(b) Show that the length of λ_a from the south pole $(0, 0, -r)$ to the north pole $(0, 0, r)$ (limit values) is equal to $\sqrt{1+a^2} \cdot \pi\, r$.

17. (*Tube*) If β is a curve in $\mathbb{R}^3$ with $0 < \kappa \leq b$, let

$$\mathbf{X}(u, v) \quad = \quad \beta(u) \;+\; \varepsilon\,(\cos b\,\mathbf{N}(u) + \sin v\,\mathbf{B}(u)\,)$$

in which $\mathbf{N}$ and $\mathbf{B}$ are the principal normal and binormal of β. Thus the v-parameter curves are circles of constant radius ε in planes orthogonal to β. Show the following:

(a) $\mathbf{X}_u \times \mathbf{X} - v = -\varepsilon(1 - \kappa\varepsilon\cos v)\big(\cos b\,\mathbf{N}(u) + \sin v\,\mathbf{B}(u)\,\big)$.

(b) If ε is small enough the function $\mathbf{X}$ is regular. Therefore $\mathbf{X}$ is a surface parametrization, and it is called a **tube** surrounding β.

(c) The unit normal vector of the tube is given by $\cos b\,\mathbf{N}(u) + \sin v\,\mathbf{B}(u)$.

(d) The Gaussian curvature is given by

$$K \quad = \quad \frac{\cos v}{\kappa(u)\varepsilon(1 - \cos v0} \ .$$

[*Hint:* Show that the Shape operator and Gaussian curvature satisfy $\mathbf{S}(\mathbf{X}_u) \times \mathbf{S}(\mathbf{X}_v) \ = \ K\,\mathbf{X}_u) \times \mathbf{X}_v$.]

Additional exercises

1.* The graph of the function $z = \log_e \cos y - \log_e \cos x$, where $|x|, |y| < \frac{1}{2}\pi$, is a piece of a surface called *Scherk's minimal surface* (see Exercise 5 from Section 5.5 of O'Neill). Prove that this is a minimal surface and its Gaussian curvature is given by $-e^{2z}/(e^{2z}\sin^2 x + 1)^2$.

2. Consider the surface S defined by the equation $z = f(u, v)$. Prove that S is minimal if and only if

$$f_{uu}\left(1 + f_v^2\right) \ - \ 2f_u f_v f_{uv} \ + \ f_{vv}\left(1 + f_u^2\right) \quad = \quad 0 \ .$$

The subscripts indicate (first or second order) partial derivatives with respect to the indicated variables. This partial differential equation is called the *minimal surface equation*.