Addendum to § II.3

The Inverse Function Theorem implies that locally every regular smooth curve in \( \mathbb{R}^2 \) is given by an equation \( F(x,y) = 0 \) where \( F \) is real valued with continuous partial derivatives and \( \nabla F \) is never zero.

**Theorem.** Let \( Y: (t_0-h, t_0+h) \) be a regular smooth curve (\( Y \) continuous) in \( \mathbb{R}^2 \). Then there is an open neighborhood \( U \) of \( Y(t_0) \) in \( \mathbb{R}^2 \), an \( h' < h \), and a smooth real valued function \( F(x,y) \) on \( U \) such that

(i) \( \nabla F \) is never zero on \( U \)

(ii) If \( (x,y) \in U \), then \( (x,y) = Y(t) \) for some \( t \in (t_0-h', t_0+h') \) if and only if \( F(x,y) = 0 \).

**Proof.** Let \( N_0 \) be obtained by a 90° counterclockwise rotation of \( Y(t_0) \):

\[
N_0 = (-y'(t_0), x'(t_0)).
\]

Define \( G(u,v) \) on the vertical strip of all \((u,v)\) with \( u \) in \((t_0-h, t_0+h)\):

\[
G(u,v) = Y(u) + vN_0
\]
Then \( G(u,v) \) has derivative matrix
\[
D G(u,v) = \begin{pmatrix}
x'(u) & -y'(t_0) \\
y'(u) & x'(t_0)
\end{pmatrix}
\]

We have \( \det G(t_0,0) = \begin{vmatrix} x'(t_0) & -y'(t_0) \\ y'(t_0) & x'(t_0) \end{vmatrix} = \begin{vmatrix} y'(t_0) \\ x'(t_0) \end{vmatrix}^2 > 0 \), so

by the Inverse Function Theorem there are open neighborhoods \( U \) of \( x(t_0) = G(t_0,0) \) and \( W \) of \( (t_0,0) \) such that \( G \) maps \( W \) to \( U \) in a 1-1 onto fashion and the inverse \( K: U \to W \)
\[
K(w_1,w_2) = (k_1(w_1,w_2), k_2(w_1,w_2)) \text{ has cont. first partials in each coord.}
\]

We can cut down \( U \) to \( W \) so that \( W \) is the set of all \((w_1,w_2)\) satisfying \( t_0 - \varepsilon < w_1 < t_0 + \varepsilon \), \( 0 < \varepsilon \leq h \)

If we let \( F = k_2 \) then \( \nabla F \) is never zero and
\[
k_2(w_1,w_2) = 0 \iff (w_1,w_2) = \gamma(k_1(\overrightarrow{w_2}))
\]

* true because the Jacobian
\[
\frac{\partial (k_1, k_2)}{\partial (w_1, w_2)} \neq 0 \text{ on } U.
\]