Addendum to § II.3

The Inverse Function Theorem implies that LOCALLY every regular smooth curve in $\mathbb{R}^2$ is given by an equation $F(x, y)=0$ where F is real valued with continuous partial derivatives and ∇F is never zero.

<u>Theorem</u> Let $\gamma : (t_0-h, t_0+h)$ $\boxed{\text{OPEN INTERVAL}}$ be a regular smooth curve (γ' continuous) in $\mathbb{R}^2$. Then there is an open neighborhood U of $\gamma(t_0)$ in $\mathbb{R}^2$, ~~and~~ an h' such that $0 < h' \leq h$, and a smooth real valued function $F(x,y)$ on U such that

(i) ∇F is never zero on U

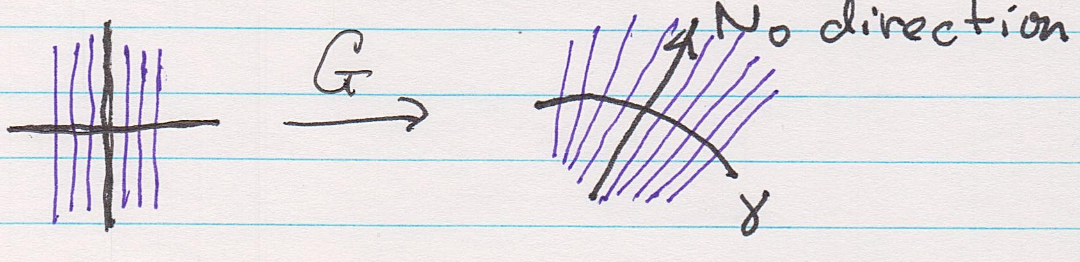
(ii) If $(x,y) \in U$, then $(x,y) = \gamma(t)$ for some t in $(t_0-h'; t_0+h') \iff F(x,y)=0$.

<u>Proof</u> Let N_0 be obtained by a $90°$ counter-clockwise rotation of $\gamma'(t_0)$:

$$N_0 = (-y'(t_0), x'(t_0)).$$

Define $G(u,v)$ on the vertical strip of all (u,v) with u in (t_0-h, t_0+h):

$$G(u,v) \equiv \gamma(u) + vN_0$$



Then $G(u,v)$ has derivative matrix

$$DG(u,v) = \begin{pmatrix} x'(u) & -y'(t_0) \\ y'(u) & x'(t_0) \end{pmatrix}$$

$\boxed{\text{Hence coords of } G \text{ have cont. partials}}$

We have $\det G(t_0, 0) = \begin{vmatrix} x'(t_0) & -y'(t_0) \\ y'(t_0) & x'(t_0) \end{vmatrix} =$

$$|\gamma'(t_0)|^2 > 0, \text{ so}$$

by the Inverse Function Theorem there ~~∃~~ are open neighborhoods U of $\gamma(t_0) = G(t_0, 0)$ and W of $(t_0, 0)$ such that G maps W to U in a 1-1 onto fashion and the inverse $K: U \to W$

$K(u_1, u_2) = (k_1(u_1,u_2),\ k_2(u_1,u_2))$ has cont. first partials in each coord.

We can cut down U & W so that W is the set of all (w_1, w_2) satisfying $\begin{cases} t_0 - \varepsilon < w_1 < t_0 + \varepsilon & 0 < \varepsilon \le h \\ -\delta < w_2 < \delta \end{cases}$

If we let $F = k_2$ then ∇F is never zero* and

$$k_2(\overset{u_1,u_2}{\cancel{w_1,w_2}}) = 0 \iff (\overset{u_1,u_2}{\cancel{w_1,w_2}}) = \gamma(k_1(\overset{u_1,u_2}{\cancel{w_1,w_2}}))$$

this runs over all t in $(t_0 - \varepsilon, t_0 + \varepsilon)$. ∎

* true because the Jacobian ~~of~~

$$\frac{\partial(k_1, k_2)}{\partial(u_1, u_2)} \neq 0 \text{ on } U.$$