4. [*25 points*] Let N be the 3×3 matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} .$$

Compute the exponential matrix $\exp tN$ where t is a real number. [*Hint:* Compute the powers of N and show that $0 = N^3 = N^4 = \cdots$.]

SOLUTION

Following the hint, compute the next two powers N^k directly using the definition of matrix multiplication:

$$N^2 \quad = \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \qquad N^3 = 0.$$

Since $N^3 = 0$ for all $k \geq 4$ we have

$$N^k \quad = \quad N^3 \cdot N^{k-3} \quad = \quad 0 \cdot N^{k-3} \quad = \quad 0 \, .$$

This means that all terms of degree ≥ 3 in the exponential series

$$\exp(tN) \quad = \quad \sum_k \frac{t^k}{k!} \, N^k$$

are zero, so that the series collapses to the finite sum

$$I \; + \; t\,N \; + \; \frac{t^2}{2} \, N^2$$

and we can now use the previously given or derived information about the powers of N to compute $\exp(tN)$ explicitly as a function of t:

$$\exp(tN) \quad = \quad \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{1}{2}\,t^2 & t & 1 \end{pmatrix}$$