CURVATURE FOR A HYPERBOLA

Look at the graph of $y = 1/x$ where $x > 0$.

Parametric equations $\gamma(t) = (t, 1/t, 0)$

$\gamma'(t) = (1, -1/t^2, 0)$ $\quad |\gamma'(t)| = \sqrt{1 + 1/t^4}$

$\gamma''(t) = (0, 2/t^3, 0)$

$$\gamma' \times \gamma'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1/t^2 & 0 \\ 0 & 2/t^3 & 0 \end{vmatrix} = (0, 0, 2/t^3).$$

$$\kappa(t) = \frac{2/t^3}{(1+1/t^4)^{3/2}} = \frac{2}{(t^2)^{3/2}(1+1/t^4)^{3/2}} =$$

$$\frac{2}{(t^2 + 1/t^2)^{3/2}}.$$

maximum κ?
minimum κ?
$\lim_{t\to\infty} \kappa(t) = ?$

- $\lim_{t\to\infty} \frac{2}{(t^2+1/t^2)^{3/2}} = 0$, so the curve "flattens out" as $t \to \infty$.

- Set $\kappa'(t) = 0$ and solve for t. <u>Better</u>: Solve $\frac{d}{dt}\left(t^2 + \frac{1}{t^2}\right) = 0$ and solve for t ($\max \kappa(t) \Leftrightarrow \min(t^2 + 1/t^2)$)

Note that the curve also flattens out as $t \to 0$.

This is true because a $\left\{\begin{matrix}\text{maximum}\\ \text{minimum}\end{matrix}\right\}$ for $\dfrac{2}{(t^2+1/t^2)^{3/2}} \Longleftrightarrow$

$\left\{\begin{matrix}\text{minimum}\\ \text{maximum}\end{matrix}\right\}$ for (t^2+1/t^2).

How to maximize or minimize the latter?

Solve $\dfrac{d}{dt}(t^2+1/t^2)=0$; get $t=1$ (since $t>0$).

This turns out to be a minimum for $\kappa(t)$:

$$\frac{d^2}{dt^2}\left(t^2+1/t^2\right)\Big|_{t=1} = 2+6/t^4\Big|_{t=1} > 0.$$

$\Rightarrow$ t^2+1/t^2 has a minimum at $t=1$.

The maximum curvature is therefore

$$\frac{2}{2^{3/2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},$$

and it occurs at the vertex $(1,1)$ of the hyperbola.