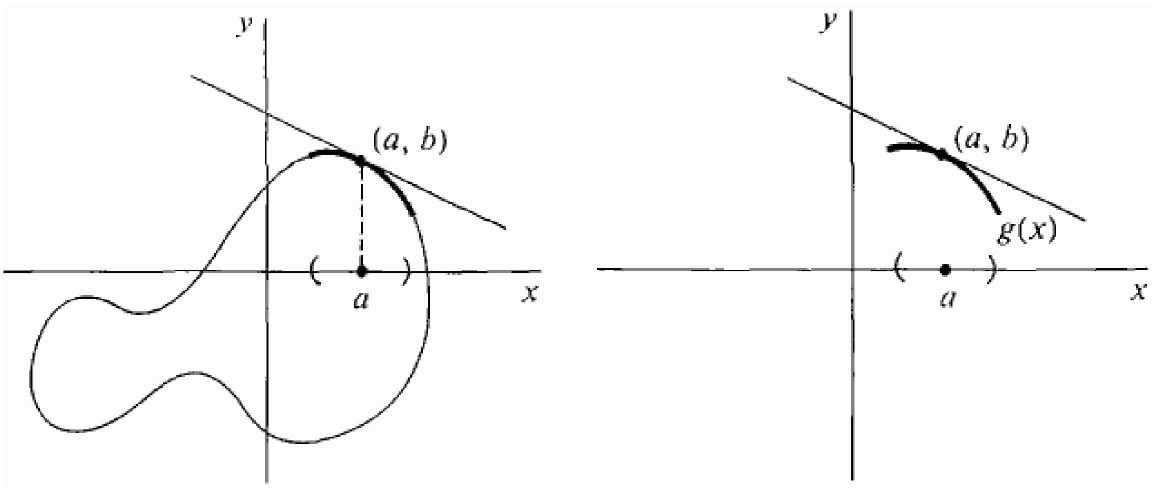
The Implicit Function Theorem

The following drawings depict the hypotheses and conclusions of the ***Implicit Function Theorem*** for a real valued function of two variables $F(x, y)$ with continuous partial derivatives; the curve in the drawing on the left represents the set of all points in the plane where $F(x, y) = 0$, and only a small piece of that curve appears in the drawing on the right.



(***Source:*** **http://www.vias.org/calculus/11_partial_differentiation_06_01.html**)

Standard formulas from multivariable calculus imply that *the gradient vector of* F *at the point* (a, b) *is perpendicular to the tangent line to the curve at* (a, b), so both of the gradient's coordinates (hence both partial derivatives of F) are nonzero at (a, b). Thus the Implicit Function Theorem implies that there is a continuously differentiable function $g(x)$ defined on a small interval centered at a such that (i) $g(x) = b$, (ii) in some suitably small open neighborhood W of (a, b) the graph of g is equal to the set of all points (x, y) in W satisfying the equation $F(x, y) = 0$. Note that if we take too large of an open neighborhood — for example, the entire vertical strip over the interval centered at the point a — then there may be points that satisfy $F(x, y) = 0$ but do not belong to the graph of g. Also note that the interval must be small enough that there are no points in the vertical strip have horizontal gradient vectors (which geometrically correspond to vertical tangent lines where two or more solutions of $F(x, y) = 0$ converge); one can always do this when the second coordinate of $\nabla F(a, b)$ is nonzero.