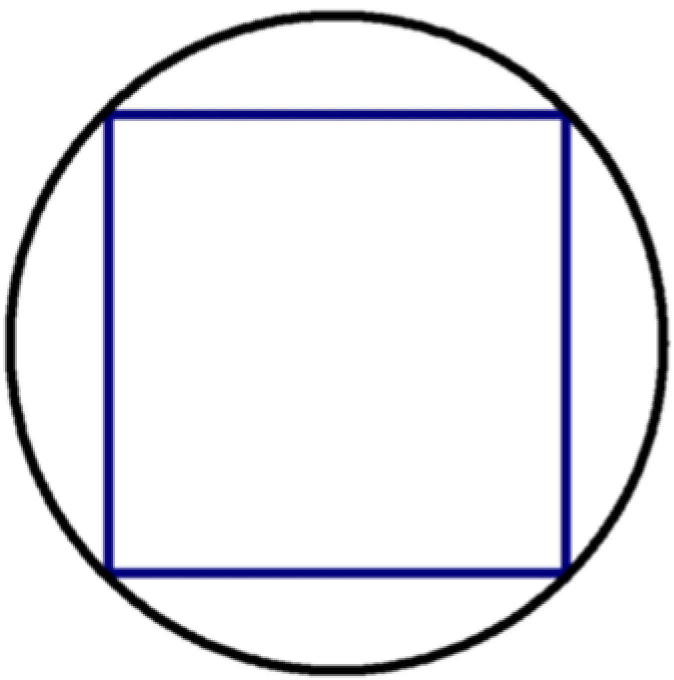
Comparing square and round neighborhoods of a point

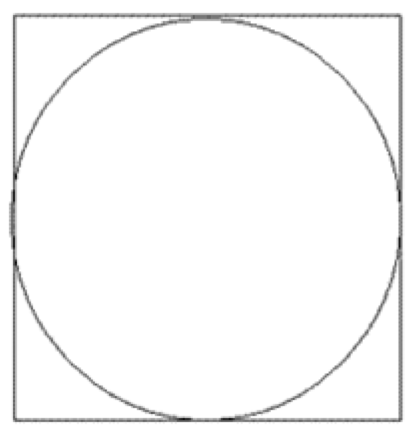
If we are given a point $\mathbf{p} = (a, b)$ in the coordinate plane, then it is often useful to talk about its neighborhoods. One type of neighborhood is the ***open square neighborhood*** of radius $h > 0$, which consists of all $\mathbf{v} = (x, y)$ such that both $|x - a|$ and $|y - b|$ are less than h (in more advanced contexts one frequently uses lower case Greek letters such as δ or ε instead of h). This set is the interior of a square whose edges have length $2h$ and whose center is (a, b). Another basic type of neighborhood is the ***open disk neighborhood*** of radius $r > 0$, which is the set of all points $\mathbf{v} = (x, y)$ such that the usual Euclidean distance

$$d(\mathbf{v}, \mathbf{p}) = \sqrt{(x-a)^2 + (y-b)^2}$$

is less than r. As indicated in Section **I.0** of the course notes, the following relationship between these two different types of neighborhoods is fundamentally important:

Given $r > 0$ there is some $h > 0$ such that the open disk neighborhood of $\mathbf{p}$ of radius r contains a square neighborhood of radius h. Conversely, given $h > 0$ there is some $s > 0$ such that the square open square neighborhood of $\mathbf{p}$ of radius h contains an open disk neighborhood of radius s.





(***Sources*: http://eldar.mathstat.uoguelph.ca/.../POW/P006.html , http://www.askmehelpdesk.com/attachments/math-sciences/4562d1190568767-aread-circle-inscribed-square-area-small-.gif**)

The drawings above illustrate the comparison statement(s) for open square and open disk neighborhoods.

In fact, if we are given the open disk neighborhood of radius $\boldsymbol{r}$, then as suggested in the first figure we can take $\boldsymbol{h}$ to be $\boldsymbol{r}/\sqrt{\mathbf{2}}$, and if we are given the open square neighborhood of radius $\boldsymbol{h}$, then as suggested in the second figure we can take $\boldsymbol{s}$ to be $\boldsymbol{h}$.

How do we actually PROVE these assertions?

In order to make the discussion mathematically complete, we need to verify that the preceding statements are correct using algebraic and analytic arguments.

First, suppose we start with a disk of radius $\boldsymbol{r}$, and as before suppose that $\boldsymbol{h}$ equals $\boldsymbol{r}/\sqrt{\mathbf{2}}$. If $(\boldsymbol{x}, \boldsymbol{y})$ is in the open square neighborhood of radius $\boldsymbol{h}$ centered at $(\boldsymbol{a}, \boldsymbol{b})$, then by definition we know that $|\boldsymbol{x} - \boldsymbol{a}|$ and $|\boldsymbol{y} - \boldsymbol{b}|$ are less than $\boldsymbol{r}/\sqrt{\mathbf{2}}$, and therefore we have

$$\boldsymbol{d}(\mathbf{v}, \mathbf{p}) \;=\; \sqrt{(\boldsymbol{x}-\boldsymbol{a})^2 + (\boldsymbol{y}-\boldsymbol{b})^2} \;<\; \sqrt{\frac{\boldsymbol{r}^2}{\mathbf{2}} + \frac{\boldsymbol{r}^2}{\mathbf{2}}} \;=\; \boldsymbol{r}$$

which is what we wanted to prove. Similarly, if we start with the square of radius $\boldsymbol{h}$ and take a point $(\boldsymbol{x}, \boldsymbol{y})$ is in the open disk neighborhood of radius $\boldsymbol{h}$ centered at $(\boldsymbol{a}, \boldsymbol{b})$, then by definition we know that $\boldsymbol{h} \;>\; \sqrt{(\boldsymbol{x}-\boldsymbol{a})^2 + (\boldsymbol{y}-\boldsymbol{b})^2}$ and the right hand side is greater than or equal to both $|\boldsymbol{x} - \boldsymbol{a}|$ and $|\boldsymbol{y} - \boldsymbol{b}|$. Therefore the point $(\boldsymbol{x}, \boldsymbol{y})$ lies in the open square neighborhood of radius $\boldsymbol{h}$ centered at $(\boldsymbol{a}, \boldsymbol{b})$, which proves the second comparison statement for square and disk neighborhoods.

<u>**Generalizations to higher dimensions.**</u> Similar considerations apply in three (or more) dimensions. Specifically, in three dimensions we can define the ***open cube neighborhood*** of $\mathbf{p} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ of radius $\boldsymbol{h} > \mathbf{0}$ to be all $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ such that each of $|\boldsymbol{x} - \boldsymbol{a}|$, $|\boldsymbol{y} - \boldsymbol{b}|$ and $|\boldsymbol{z} - \boldsymbol{c}|$ is less than $\boldsymbol{h}$, and the ***open disk neighborhood*** of radius $\boldsymbol{r} > \mathbf{0}$, which is the set of all points such that the usual Euclidean distance is less than $\boldsymbol{r}$. In this case, if we start with $\boldsymbol{r}$ then we can take $\boldsymbol{h}$ to be $\boldsymbol{r}/\sqrt{\mathbf{3}}$, and if we with $\boldsymbol{h}$, then we can take $\boldsymbol{s} = \boldsymbol{h}$.

If we want to consider even higher dimensions, then the analog of a square or cube neighborhood is a ***hypercube neighborhood***, and the defining conditions are that the absolute values of all coordinate differences are less than $\boldsymbol{h}$. There are also corresponding comparison statements: If we start with $\boldsymbol{r}$ then we can take $\boldsymbol{h}$ to be $\boldsymbol{r}/\sqrt{\boldsymbol{n}}$, where $\boldsymbol{n}$ is the dimension in which we are working, and if we with $\boldsymbol{h}$, then we can take $\boldsymbol{s} = \boldsymbol{h}$.