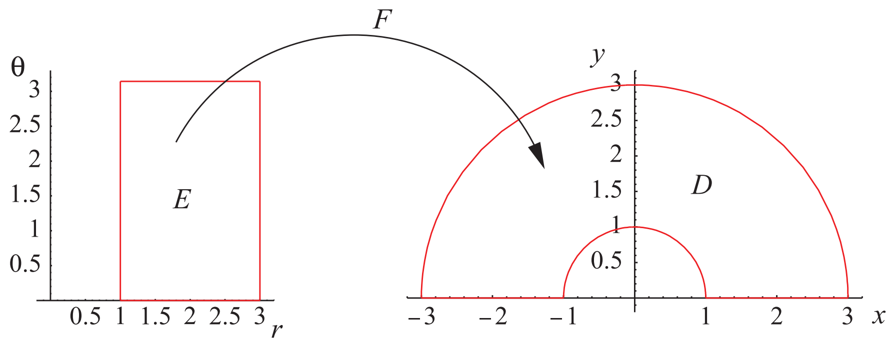

Figure 3.7.6 Polar coordinates map $[1,3] \times [0,\pi]$ to top half of an annulus

We wish to evaluate

$$\int\int_D e^{-(x^2+y^2)} dxdy.$$

Under the polar coordinate change of variables

$$x = r\cos(\theta)$$

and

$$y = r\sin(\theta),$$

the annular region D corresponds to the closed rectangle

$$E = \{(r,\theta) : 1 \le r \le 3, 0 \le \theta \le \pi\},$$

as illustrated in Figure 3.7.6. Moreover, $x^2 + y^2 = r^2$ and, as we saw in the previous example,

$$\left|\det \frac{\partial(x,y)}{\partial(r,\theta)}\right| = r.$$

Hence

$$\begin{aligned}\int\int_D e^{-(x^2+y^2)} dxdy &= \int\int_E re^{-r^2} drd\theta \\ &= \int_1^3 \int_0^\pi re^{-r^2} d\theta dr \\ &= \int_1^3 \pi re^{-r^2} dr \\ &= -\frac{\pi}{2} e^{-r^2}\Big|_1^3 \\ &= \frac{\pi}{2}(e^{-1} - e^{-9}).\end{aligned}$$

Note that in this case the change of variables not only simplified the region of integration, but also put the function being integrated into a form to which we could apply the Fundamental Theorem of Calculus.