Summary of approaches to the Second Fundamental Form

The Second Fundamental Form measures the change in the normal direction to the tangent plane as one moves from point to point on a surface Σ, and its definition requires a specific choice of a smooth vector valuked function $\mathbf{N}$ on Σ such that for each point $\mathbf{p}$ on the surface the vector $\mathbf{N}(\mathbf{p})$ is a unit normal to the tangent plane for Σ at $\mathbf{p}$. As indicated in the notes, it is possible to extend $\mathbf{N}$ to a smooth function defined on some small open neighborhood of $\mathbf{p}$ in $\mathbf{R}^3$ (this uses the fact that one can locally "flatten" the surface by a smooth change of variables). The vector valued function $\mathbf{N}$ is generally called an **orientation** of the surface, and a surface together with an orientation is called an *oriented surface*.

Every surface which is given by either of

(1) a smooth regular 1–1 parametrization $\sigma(u, v)$,

(2) the level set of some smooth function $F(x, y, z) = 0$ on which $\nabla F(x, y, z) \neq \mathbf{0}$, can be oriented fairly easily.

In the first case, one can take $\mathbf{N}$ to be the unit vector pointing in the same direction as the partial derivative cross product $\sigma_u \times \sigma_v$, and in the second case one can take $\mathbf{N}$ to be the unit vector pointing in the same direction as the gradient ∇F. Locally there are exactly two choices of orientation and they differ by a $\pm$ sign. Most basic examples of surfaces turn out to be orientable, but the Möbius strip is an example of a surface which is not.

If our surface is given by a regular smooth parametrization $\sigma(u, v)$ which is 1–1 (the latter is always true locally), then the classical definition of the Second Fundamental Form is a formal dot product of differential expressions:

$$-d\mathbf{N} \cdot d\sigma \;=\; -\,(\,\mathbf{N}_u\, du + \mathbf{N}_v\, dv\,) \cdot (\,\sigma_u\, du + \sigma_v\, dv\,) \;=$$
$$-\,(\,(\mathbf{N}_u \cdot \sigma_u)\, du\, du \;+\; (\mathbf{N}_u \cdot \sigma_v + \mathbf{N}_v \cdot \sigma_u)\, du\, dv \;+\; (\mathbf{N}_v \cdot \sigma_v)\, dv\, dv\,)$$

This might look a little suspicious because it involves symbols like differentials, but it turns out that such notation can be justified using the language of differential forms (in particular, this is not like a heuristic derivation of some physical or engineering formula containing steps like "take a flat metal sheet of infinitesimal thickness dx").

Even in relatively simple cases the computation of the partial derivatives $\mathbf{N}_u$ and $\mathbf{N}_v$ can be very messy, so it is useful to have alternate formulas for the computation of the coefficients of $du\, du$, $du\, dv$, and $dv\, dv$. For parametrized surfaces the formulas

$$\mathbf{N}_u \cdot \sigma_u \;=\; -\,\mathbf{N} \cdot \sigma_{u,u}\,, \quad \mathbf{N}_u \cdot \sigma_v \;=\; -\,\mathbf{N} \cdot \sigma_{u,v} \;=\; \mathbf{N}_v \cdot \sigma_u\,, \quad \mathbf{N}_v \cdot \sigma_v \;=\; -\,\mathbf{N} \cdot \sigma_{v,v}$$

are extremely useful.

Shape operators

In some situations it is convenient to describe the Second Fundamental Form of a regular geometric surface in a manner which does not depend upon the choice of a local parametrization. There is a corresponding description of the First Fundamental Form; namely, if $T_p(\Sigma)$ denotes the 2-dimensional space of tangent vectors at the point p of the surface Σ, then the first fundamental form at p is merely the restriction of the standard dot or inner product on $\mathbf{R}^3$ to the subspace

$T_p(\Sigma)$. The corresponding description of the Second Fundamental Form at p involves a somewhat different function $\varphi_p(v, w)$ which is defined for ordered pairs of vectors in $T_p(\Sigma)$, takes values in the real numbers, and satisfies the following analogs of inner product identities:

$$\varphi_p(v, w + w') \;=\; \varphi_p(v, w) + \varphi_p(v, w')$$
$$\varphi_p(v + v', w) \;=\; \varphi_p(v, w) + \varphi_p(v', w)$$
$$\varphi_p(v, c\,w) \;=\; c\,\varphi_p(v, w) + \varphi_p(c\,v, w)$$
$$\varphi_p(v, w) \;=\; \varphi_p(w, v)$$

In these identities v, v', w, w' are vectors in $T_p(\Sigma)$ and c is a scalar. A mapping φ which has these properties is called a **symmetric bilinear form**, and hence the First and Second Fundamental Forms of a surface are families of symmetric bilinear forms for the 2-dimensional vector spaces $T_p(\Sigma)$.

The two fundamental forms $\mathbf{I}_p$ and $\mathbf{II}_p$ are related by a family of linear transformations

$$\$_p : T_p(\Sigma) \;\longrightarrow\; T_p(\Sigma)$$

collectively known as the *shape operator* or *Weingarten map*. Section IV.2 of the notes describes these maps in a manner that does not depend upon a choice of local parametrization. However, these maps can be described very explicitly if one is given a 1–1 local parametrization σ as follows: Assume that the local parametrization is defined on an open region U, and write the parametrization as $\sigma(u_1, u_2)$; we shall use subscripts to denote ordinary and higher order partial derivatives of σ with respect to appropriate variables (these partial derivatives are 3-dimensional vector valued). In this notation the partial derivatives $\sigma_1(p)$ and $\sigma_2(p)$ will be a basis for $T_p(\Sigma)$.

Since a linear transformation on a finite-dimensional vector space is uniquely determined by its values on a basis, it is only necessary to define $\$_p$ for $\sigma_1(p)$ and $\sigma_2(p)$. Specifically, the definition is

$$\$_p\big(\sigma_1(p)\big) \;=\; \mathbf{N}_1(p)\,, \qquad \$_p\big(\sigma_2(p)\big) \;=\; \mathbf{N}_2(p)\,.$$

Here $\mathbf{N}_j$ denotes a partial derivative of $\mathbf{N}$ viewed as a function on the open region U via the parametrization $\sigma : U \to \Sigma$. Although it might not be immediately obvious whether or not these partial derivatives lie in $T_p(\Sigma)$, this fact follows because if we partial differentiate $|\mathbf{N}|^2 = 1$ then we obtain the identities $2\,\mathbf{N}_j \cdot \mathbf{N} = 0$ for $j = 1, 2$; since $T_p(\Sigma)$ is the set of all vectors in $\mathbf{R}^3$ which are perpendicular to $\mathbf{N}$, this proves that $\mathbf{N}_1$ and $\mathbf{N}_2$ lie in $T_p(\Sigma)$ so that we can define $\$_p$ in the desired manner.

The Second Fundamental Form is then given by

$$\mathbf{II}_p(v, w) \;=\; \mathbf{I}_p(\$_p(v), w)\,.$$

All of the properties except $\mathbf{II}_p(v, w) = \mathbf{II}_p(w, v)$ follow immediately from this definition. Straightforward algebraic considerations imply that this last property will hold if we can show that

$$\mathbf{I}_p(\$_p(\sigma_1), \sigma_2) \;=\; \mathbf{I}_p(\$_p(\sigma_2), \sigma_1)\,.$$

But this identity is a consequence of the following sequence of equations:

$$\mathbf{I}_p(\$_p(\sigma_1), \sigma_2) \;=\; \mathbf{I}_p((\mathbf{N}_1), \sigma_2) \;=\; -\,\mathbf{I}_p((\mathbf{N}), \sigma_{12}) \;=\; \mathbf{I}_p((\mathbf{N}_2), \sigma_1) \;=\; \mathbf{I}_p(\$_p(\sigma_2), \sigma_1)$$