Computing the Second Fundamental Form

$\vec{X}_{i} :=$ partial derivative with respect to ith variable

Plane $z = 0$

$\vec{X}(u,v) = (u, v, 0)$.

$\vec{X}_1(u,v) = (1,0,0)$ $\vec{X}_2(u,v) = (0,1,0)$

$\vec{X}_1 \times \vec{X}_2 = (0,0,1)$ Since this is a unit vector, we may take this to be $\vec{N}$.

SFF coefficients.

$e = \vec{N} \cdot \vec{X}_{11} = \vec{N} \cdot \vec{0} = 0$

Likewise for f and g. So we have

$SFF = 0$.

Cylinder $x^2 + y^2 = 1$. $\vec{X} = (\cos u, \sin u, v)$.

$\vec{X}_1 = (-\sin u, \cos u, 0)$ $\vec{X}_2 = (0,0,1)$

So $\vec{X}_1 \times \vec{X}_2 = (\text{calculations}) = (-\cos u, -\sin u, 0)$.

This is again a unit vector, so we can take this to be $\vec{N}$ (inward normal in this case).

In both cases $\vec{X}_1$ and $\vec{X}_2$ are orthonormal so that they have the same FFF:

$$E = \vec{X}_1 \cdot \vec{X}_1 = 1 \quad , \quad F = \vec{X}_1 \cdot \vec{X}_2 = 0, \quad G = \vec{X}_2 \cdot \vec{X}_2 = 1$$

However, if we compute the SFF for the cylinder, here is what we get:

$$\vec{X}_{11} = (-\cos u, -\sin u, 0)$$

$$\vec{X}_{12} = \vec{X}_{22} = 0.$$

Hence $f = g = 0$ immediately and

$$e = (-\cos u, -\sin u, 0) \cdot (-\cos u, -\sin u, 0) = 1$$

so that $SFF = du\,du$, nonzero.

ANOTHER EXAMPLE

Consider the cone $x^2 + y^2 = z^2$, $z > 0$,

So that one has a parametrization

$$\vec{X}(u,v) = (v\cos u, v\sin u, v)$$

In this case

$\vec{X_1} = (-v \sin u, v \cos u, 0)$

$\vec{X_2} = (\cos u, \sin u, 1)$.

Here are the coefficients of the FFF:

$E = v^2 \qquad F = 0 \qquad G = 2$

To compute the SFF, first consider

$$\Omega = \vec{X_1} \times X_2 = \begin{vmatrix} i & j & k \\ -v \sin u & v \cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix} =$$

$(v \cos u, v \sin u, -v)$. Therefore

$$N = \frac{1}{|\Omega|} \Omega = \frac{1}{v\sqrt{2}} \Omega = \frac{1}{\sqrt{2}} (\cos u, \sin u, -1).$$

Now $\vec{X_{11}} = (-v \cos u, -v \sin u, 0)$

$\vec{X_{12}} = (-\sin u, \cos u, 0)$

$\vec{X_{22}} = (0, 0, 0)$

These lead to

$$e = -\frac{v}{\sqrt{2}}, \quad f = 0, \quad g = 0.$$

Recommended exercise

What happens if we take a cylinder of radius $r > 0$ with parametrization $\left(r\cos\frac{u}{r}, r\sin\frac{u}{r}, v\right)$ or a cone of the form $x^2 + y^2 = a^2 z^2$, where $a > 0$, with parametrization

$$(av\cos u, av\sin u, v)?$$