A signed curvature example

GENERALITIES: $\gamma(s)$ regular smooth curve in the plane, "arc length" parametrization s.

$J : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ rotation through 90° counter-clockwise, so that $J(x,y) = (-y, x)$.

$|\gamma'(s)| = 1$ (arc length parametrization).

$T(s) = \gamma'(s)$ is the unit tangent

$N(s) = J(T(s))$, a unit vector $\perp T(s)$.

As before $T'(s) \perp T(s)$. Let the signed curvature $k(s)$ be given by

$$T'(s) = k(s) \cdot N(s), \text{ or equivalently}$$

$$k(s) = T'(s) \cdot N(s)$$

COMPUTATIONAL TECHNIQUES: How to compute signed curvature for an arbitrary parametrization $\gamma(t)$.

Key example: graph of x^3:

$$\gamma(t) = (t, t^3).$$

As usual, let s = arc length, so $\gamma'(t) = s(t)\cdot T(t)$. Then

$$k = \frac{dT}{ds}\cdot N = \frac{T'(t)}{s'(t)}\cdot N(t) =$$

$$\frac{T'(t)}{s'(t)}\cdot J(N(t)).$$

For the key example, $\gamma'(t) = (1, 3t^2)$

$$\text{let } y^2 = |\gamma'|^2 = 1 + 9t^4.$$

Then $T(t) = \frac{1}{y}(1, 3t^2)$, $N(t) = \frac{1}{y}(-3t^2, 1)$

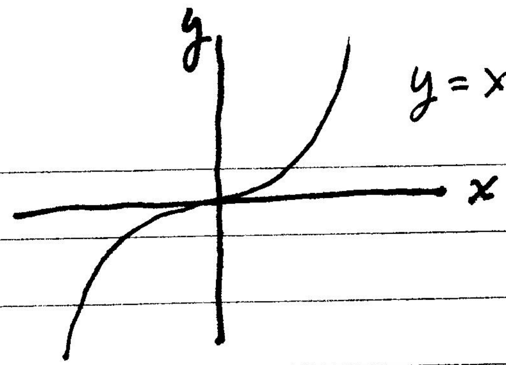
$$T'(t) = \left(-\frac{1\cdot(36t^3)}{2y^3},\ -\frac{3t^2(36t^3)}{2y^3} + \frac{6t}{y}\right)$$

$$= \frac{1}{2y^3}\left(-36t^3,\ -108t^5 + 12t\,y^2\right) =$$

$$\frac{1}{2y^3}\left(-36t^3, -108t^5 + 12t(1+9t^4)\right) =$$

$$\frac{1}{2y^3}\left(-36t^3, 12t\right) = \frac{12t}{2y^3}(-3t^2, 1)$$

$$\text{So } k(t) = \frac{12t}{2y^2} = \frac{6t}{y^2}.$$



Notice that the signed curvature is positive if $x > 0$ and negative if $x < 0$.

Visually, this change of sign reflects the fact that the curve crosses the tangent line of the curve at $(0,0)$. For $x > 0$ the curve lies on one side of the tangent line, while for $x < 0$ the curve lies on the other side.