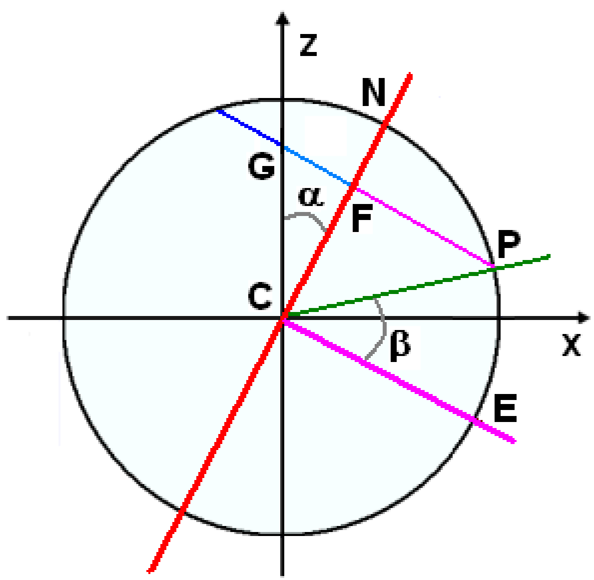
Derivation of the solstice formula

The goal is to derive the formula for the length of the longest day which appears in the document http://math.ucr.edu/~res/solstice/solstice.pdf. In this discussion β represents the latitude, and α is the angle 23.5 degrees (approximately the northernmost latitude that the sun reaches). We assume that β lies between 0 (the latitude of the equator) and 66.5 degrees North (the approximate latitude of the Arctic Circle).

$$\text{Day Length} = 24 \cdot \left(1 - \frac{1}{\pi} \text{Arc}\cos\left(\tan\alpha \ \tan\beta\right)\right)$$

A table of values for this function in increments of one degree (less in higher latitudes) is given in the document http://math.ucr.edu/~res/solstice/solstice-table.pdf.

We begin with a drawing to describe the situation. In this drawing we assume that the $y-$axis is pointing upward from the surface of the document, and the drawing represents the planar slice defined by $y = 0$.

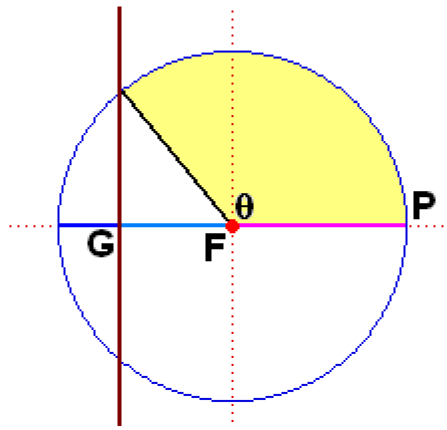


We assume that the earth corresponds to the sphere with equation $x^2 + y^2 + z^2 = 1$ in coordinate $3-$space and that the sun is located at some point of the form $(d, 0, 0)$ where d is much greater than 1. Then the points on the sphere illuminated by the sun lie in the half – space of all points whose $x-$coordinate is nonnegative.

The circle $\Gamma(\beta)$ of latitude β (North) goes through the point **P**, it is centered at **F** such that **PF** is perpendicular to **NC**, and the plane of the circle $\Gamma(\beta)$ is perpendicular to **F** at the latter point. This circle intersects the plane with equation $x = 0$ in exactly two points, and if β is positive then the major arc determined by these points has some angular measure 2θ. It follows that the amount of daytime at latitude β (in hours) is equal to $24 \cdot \theta/\pi$.

Clearly we need to find a formula for θ in terms of α and β. One step in this process is to find the lengths **|FP|** and **|GF|** of the segments **[FP]** and **[GF].** Elementary trigonometry implies **|FP|** = $\cos\ \beta$. Similarly, the length β is equal to $\sin\beta$, and therefore it follows that **|GF|** = **|CF|** $\tan\alpha = \sin\beta\tan\alpha$.

Next, we need to express **|GF|** in terms of θ. In order to do this we need to take a different view of the circle containing the arc of latitude β; the drawing below depicts this arc in the plane containing it, looking perpendicularly downward at the plane.



It follows that **|FG|** is equal to **|FP|**, which is $\cos\ \beta$, times the absolute value of $\cos\ \theta$, which is just $\cos\ (\pi-\theta)$. Therefore we have the following equation:

$$\cos\ (\pi-\theta) \ = \ |FG|/|FP| \ = \ \sin\beta\tan\alpha/\cos\ \beta \ = \ \tan\alpha\ \tan\beta$$

Therefore the total fraction of daylight time at the given latitude β is equal to

$$\frac{\theta}{\pi} = \frac{\pi - (\pi - \theta)}{\pi} = 1 - \frac{1}{\pi}(\pi - \theta)$$

and by the previous formula we know that $\pi-\theta$ is equal to $\mathrm{Arc}\cos\left(\tan\alpha\ \tan\beta\right)$. To find the total number of daylight hours, we need to multiply the resulting expression

$$1 - \frac{1}{\pi}\,\mathrm{Arc}\cos\left(\tan\alpha\ \tan\beta\right)$$

by 24 (hours), and if we do so we obtain the formula at the beginning of the document.