Solutions to Problems in Unit I

(starred items omitted)

I.1

[We are omitting the arrows in this exc.]

1. Use the BAC–CAB rule:

$$a\times(b\times c)= b(a\cdot c)-c(a\cdot b)$$
$$b\times(c\times a)= c(b\cdot a)-a(b\cdot c)$$
$$c\times(a\times b)= a(c\cdot b)-b(c\cdot a)$$

Since $(x\cdot y)=(y\cdot x)$ for all vectors x & y, the sum of the three right hand expressions (to the right of the equals sign) is zero, so the same is true for the sum of the left hand expressions.

2. $$v\times w = v\times(u\times v)= u\underset{1}{(v\cdot v)}-v\underset{0}{(v\cdot u)}=u$$

$$w\times u=(u\times v)\times u=-u\times(u\times v)=$$
$$-u(u\cdot v)+v(u\cdot u)=v.$$

Derivation of the statement in the note

write $\vec{x}=\sum_{i=1}^{3}x_ie_i$, $\vec{y}=\sum_{j=1}^{3}y_je_j$

Then we have

$$T(\vec{x}\times\vec{y}) = \sum_{i,j} x_i y_j T(e_i \times e_j) \quad \boxed{A}$$

$$T(\vec{x})\times T(\vec{y}) = \sum_{i,j} x_i y_j T(e_i)\times T(e_j) \quad \boxed{B}$$

(expand using the distributive law for $\times$ and linearity for T)

If we know $T(e_i \times e_j) = T(e_i)\times T(e_j)$ then both of these expressions $\boxed{A}$ $\boxed{B}$ will be equal. One can check these conditions for the nine cases $i=1,2,3$ & $j=1,2,3$ individually using the facts that $\vec{u}\times\vec{v}=\vec{w}$, standard rules for manipulating cross products, and the identities in the exercise.

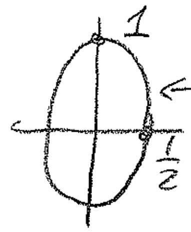
I.2

O'N-2. $\gamma(t) = \gamma(0) + \int_0^t \gamma'(u)\,du =$

$$(1,0,5) + \left(\frac{u^3}{3}, \frac{u^2}{2}, e^u\right)\Bigg|_{u=0}^{u=t} =$$

$$\left(\frac{t^3}{3}+1, \frac{t^2}{2}, e^t+4\right).$$

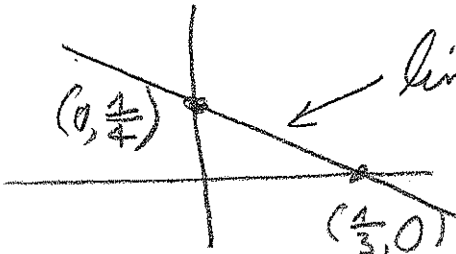
O'N-8. (a)



← ellipse

$\gamma(t) = \left(\tfrac{1}{2}\cos t, \sin t\right)$

(b)

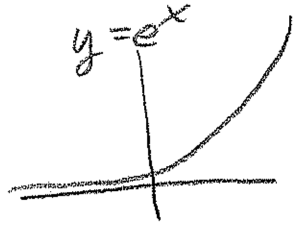


line

$\gamma(t) = \left(\frac{t}{3}, \frac{1-t}{4}\right)$

many others possible !!

(c)



$\gamma(t) = (t, e^t)$

1. $\gamma(t) = (-\sin t, \cos t) =$

$$\left(\cos\left(t+\tfrac{\pi}{2}\right), \sin\left(t+\tfrac{\pi}{2}\right)\right)$$

2. If $\alpha(t_0)$ is the (a) place where the curve is closest to the origin and $\alpha(t_0) \neq 0$, then $|\alpha(t_0)|^2$ is minimized at t_0 and hence $0 = \frac{d}{dt}\,\alpha(t)\cdot\alpha(t)$ for $t = t_0$.

But $\frac{d}{dt}\left[\alpha(t)\cdot\alpha(t)\right] = 2\alpha(t)\cdot\alpha'(t)$, so $\alpha(t_0)\cdot\alpha'(t_0)$ must be zero.

3. Given: $X = 3\cos t$ $y = 2\sin 2t = 4\sin t\cos t$

Square everything:

$x^2 = 9\cos^2 t$ $y^2 = 16\sin^2 t\cos^2 t = 16(1-\cos^2 t)\cos^2 t =$

$16\left(1-\frac{x^2}{9}\right)\left(\frac{x^2}{9}\right) =$

$\frac{16}{9}\left[\cancel{x^2} - \frac{x^4}{9}\right]$, so one choice is

$$P(x,y) = \frac{16}{9}x^2 - \frac{16}{81}x^4 - y^2.$$

4. meeting point is where $\vec{x}(t) = \vec{y}(t)$

$$\left.\begin{aligned} t^2 - 2 &= t \\ \tfrac{1}{2}t^2 - 1 &= 5 - t^2 \end{aligned}\right\}$$ usually such systems are overdetermined & have no solutions, but the second eqn is equiv. to $t^2 - 2 = 10 - 2t^2$, so we can combine the eqns to get $t^2 - 2 = 10 - 2t^2$, so that $3t^2 = 12$ or $t = \pm 2$.* The angle θ at which they meet satisfies $\cos\theta = \dfrac{\vec{x}'(t)\cdot\vec{y}'(t)}{|\vec{x}'(t)|\cdot|\vec{y}'(t)|}$ at $t = +2$.

* $\left[t^2 - 2 = t \Rightarrow -2 \text{ is not a solution}\right]$

Now $\vec{x}'(t) = (2t, t)$ and $\vec{y}'(t) = (1, -2t)$

So we have $\cos\theta = \dfrac{2t - 4t^2}{|t|\sqrt{5}\sqrt{1+4t^2}} =$

$\dfrac{2}{\sqrt{5}} \operatorname{sgn}(t) \dfrac{1-2t}{\sqrt{1+4t^2}} =$

$\dfrac{-6}{\sqrt{5}\sqrt{17}}$.

I, 3

O'N-3. $\alpha'(t) = (\sinh t \ \cosh t, 1) \Rightarrow$

$|\alpha'(t)| = \sqrt{\cosh^2 t + \sinh^2 t + 1} =$

$\sqrt{\cosh^2 t + \sinh^2 t + (\cosh^2 t - \sinh^2 t)} = \sqrt{2\cosh^2 t}$
$= \sqrt{2} \cdot \cosh t$

So arc length has $s(t) = \int_0^t \sqrt{2} \cdot \cosh u \, du =$
$\sqrt{2} \sinh t$.

O'N-4. Find values of t such that

$\alpha(t) = \begin{cases} (2, 1, 0) & t = 1 \text{ works} \\ (4, 4, \log 2) & t = 2 \text{ works} \end{cases}$

$|\alpha'(t)| = \sqrt{4 + 4t^2 + \frac{1}{t^2}} = \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}} = \frac{2t+1}{t}$

$$\text{Length} = \int_1^2 |\alpha'(u)|\,du = \int_1^2 \left(2+\frac{1}{u}\right)du =$$

$$2u + \log u \Big|_1^2 = \boxed{2 + \log 2}.$$

O'N-5. Reparametrization means we can write $t = t(s)$ and $s = s(t)$ where s and t are inverse to each other. This means, say, that $\beta_2(t(s)) = \beta_1(s)$. Both $\beta_2'(t)$ and $\beta_1'(s)$ have unit length. By the Chain Rule,

$$\frac{d\beta_2}{ds} = \frac{d\beta_2}{dt}\frac{dt}{ds} = \frac{d\beta_1}{ds}. \text{ Take lengths:}$$

$$\left|\frac{d\beta_2}{ds}\right| = 1 \cdot \frac{dt}{ds} = \left|\frac{d\beta_1}{ds}\right| = 1.$$

$\left(\left|\frac{d\beta_2}{dt}\right| = 1\right)$ This implies $\frac{dt}{ds} = 1$,

so that $t = s + \text{Constant}$. The physical meaning of ~~of~~ the Constant is that it measures a difference between the clock conventions for the parameters t and s (for example, Pacific Time vs. Eastern Time).

<u>O'N-10.</u> If $\alpha'(t) \equiv \beta'(t)$ then

$$\frac{d}{dt}\alpha - \beta = 0 \Rightarrow \alpha - \beta = \vec{c} \Rightarrow \alpha = \beta + \vec{c}.$$

<u>O'N-11</u> Skip this one.

0. Skip this one - see p.18 for a replacement.

1. P = plane $\vec{N} \cdot \vec{x} = 0$
   L = line $\vec{x}_0 + t\vec{w}$ $\quad \vec{w} \neq 0.$

$\boxed{A}$ Suppose $\vec{N} \perp \vec{u}$. <u>Claim</u> $L \subseteq P$ or $L \cap P = \phi$.

Need to show $L \cap P \neq \phi \Rightarrow L \subseteq P$.

Say $\vec{x}_0 + a\vec{u} \in P$ so that $(\vec{x}_0 + a\vec{w}) \cdot \vec{N} = 0$. Since $\vec{u} \cdot N = 0$ this means $\vec{x}_0 \cdot \vec{N} = 0$ and hence

$$(\vec{x}_0 + t\vec{u}) \cdot \vec{N} = (\vec{x}_0 \cdot N) + t(\vec{u} \cdot N) = 0 \text{ all } t.$$

$\boxed{B}$ Suppose $\vec{N}$ and $\vec{u}$ are <u>not</u> $\perp$. Then there is a basis for $\mathbb{R}^3$ consisting of $\vec{u}, \vec{v}, \vec{w}$, where $\vec{v}, \vec{w} \in P$ so that

$$\vec{v} \cdot \vec{N} = \vec{w} \cdot \vec{N} = 0.$$

Write $\vec{x}_0 = r_1\vec{u} + r_2\vec{v} + r_3\vec{w}$. We need to find t such that

$$(\vec{x}_0 + t\vec{u})\cdot\vec{N} = 0.$$

But $(\vec{x}_0 + t\vec{u})\cdot\vec{N} = r_1(\vec{u}\cdot\vec{N}) + t(\vec{u}\cdot\vec{N})$ and $(\vec{u}\cdot\vec{N}) \neq 0 \Rightarrow$ if we take $t = -r_1$, then $\vec{x}_0 + t\vec{u} \in P$ so that L and P meet.

4. (a) Cycloid: $\vec{x}'(t) = a(1-\cos t, \sin t)$

$$|\vec{x}'(t)| = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2 - 2\cos t}$$

Now $\sin\frac{1}{2}t = \sqrt{\frac{1-\cos t}{2}}$, so $|\vec{x}'(t)| = 2a\sin\frac{1}{2}t$, and therefore the arc length is

$$2a\int_0^{2\pi} \sin\tfrac{1}{2}t\,dt \underset{u=\frac{t}{2}}{=\!=\!=} 2a\int_0^{\pi} \sin u\,(2\,du) =$$

$$4a\cdot 2 = 8a.$$

(This formula has been attributed to Christopher Wren (1632–1723), who is best known for his architectural works.)

(c) Epicycle $\vec{x}'(t) = (-\sin t, \cos t) + (-\sin 4t, \cos 4t)$

$$|x'(t)| = \sqrt{2 + 2\sin t \sin 4t + 2\cos t \cos 4t} = \sqrt{2 + 2\cos 3t} = 2\left|\cos\tfrac{3}{2}t\right|.$$

So length $= 2\int_0^{2\pi} \left|\cos\tfrac{3}{2}t\right| dt$. Some care is needed because $\left|\cos\tfrac{3}{2}t\right|$ is $\begin{cases} \text{nonneg.} & 0 \le t \le \frac{\pi}{3},\ \pi \le t \le \frac{5\pi}{3} \\ \text{neg.} & \frac{\pi}{3} \le t \le \pi,\ \frac{5\pi}{3} \le t \le 2\pi \end{cases}$

The integral we want is

$$2\int_0^{\pi/3} \cos\frac{3t}{2}\,dt - 2\int_{\pi/3}^{\pi} \cos\frac{3t}{2}\,dt + 2\int_{\pi}^{5\pi/3} \cos\frac{3t}{2}\,dt - 2\int_{5\pi/3}^{2\pi} \cos\frac{3t}{2}\,dt.$$

We can calculate the individual integrals as in first year calculus, and if we do so we find that the arc length equals 8.

I.4

1. (i) Use $x = r\cos\theta$ where $r = r(\theta)$.
$y = r\sin\theta$

$$\gamma(\theta) = (r(\theta)\cos\theta,\ r(\theta)\sin\theta)$$

$$\gamma'(\theta) = r'(\theta)(\cos\theta, \sin\theta) + r(\theta)(-\sin\theta, \cos\theta)$$ } the two terms are $\perp$ and the vectors have unit length

So $|\gamma'(\theta)| = \sqrt{(r')^2 + r^2}$

(ii) $$\kappa = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3} = \frac{|\gamma' \times \gamma''|}{(r'^2 + r^2)^{3/2}}$$

So everything reduces to computing $|\gamma' \times \gamma''|$.

$$\gamma''(\theta) = \left[r''(\theta) - r(\theta)\right](\cos\theta, \sin\theta) + 2r'(\theta)(-\sin\theta, \cos\theta).$$

Add a zero 3rd coord. to γ' & γ'' in order to get 3D vectors.

Now $(\cos\theta, \sin\theta, 0) \times (-\sin\theta, \cos\theta, 0) =$
$(0, 0, 1)$

so $\gamma' \times \gamma''$ is just $(0,0,1)$ times the scalar
$2(r')^2 + -(r'' - r)r = 2(r')^2 + r^2 - r''r.$

2. (a) $\frac{ds}{dt} = |\alpha'(t)| \Rightarrow \frac{dt}{ds} = \frac{1}{|\alpha'(t)|}$

$$\frac{d^2t}{ds^2} = \frac{d}{ds}\frac{1}{|\alpha'(t(s))|} = -\frac{1}{|\alpha'|^2}\frac{d|\alpha'|}{ds} =$$

$$-\frac{1}{|\alpha'|^2}\frac{d|\alpha'|}{dt}\frac{dt}{ds} = -\frac{1}{|\alpha'|^3}\frac{d|\alpha'|}{dt}. \text{ But } \frac{d|\alpha'|}{dt} =$$

$$\frac{d}{dt}\sqrt{\alpha'\cdot\alpha'} = \frac{1}{\sqrt{\alpha'\cdot\alpha'}}\cdot\alpha'\cdot\alpha'' \Longrightarrow$$

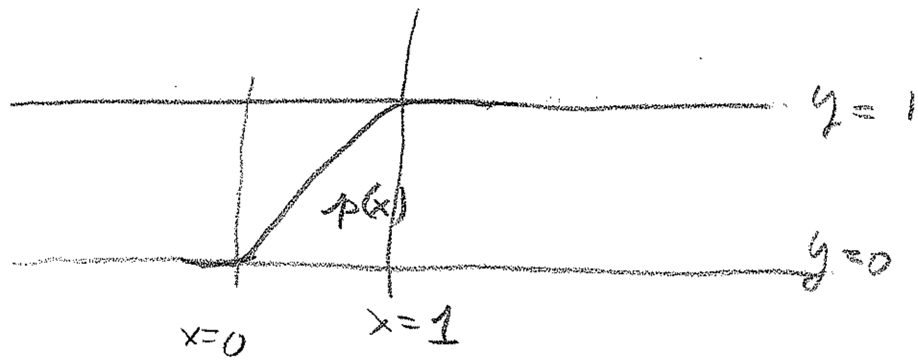
$$\frac{d^2t}{ds^2} = -\frac{1}{|\alpha'|^4}(\alpha'\cdot\alpha'').$$

(b) Done in the notes.

(c) $\tau = -\frac{dB}{ds} \cdot N$

See Lipschutz, Problem 4.19, p. 77.

4.



Need $p(0) = 0$ $\quad p(1) = 1$ $\quad \deg p = 5$

$p'(0) = p'(1) = 0$

curvature is zero when $t = 0, 1$

$p(x) = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex \Rightarrow p(0) = 0$

$1 = A + B + C + D + E = p(1)$

$0 = p''(0) = E$

$0 = p'(1) = 5A + 4B + 3C + 2D + \cancel{E}$

$k = \frac{|y''|}{(1 + (y')^2)^{3/2}}$, so $k = 0 \Leftrightarrow y'' = 0$.

$\Rightarrow \quad 0 = k(0) = 2D$

want $\quad 0 = k(1) = 20A + 12B + 6C$.

Here is a summary of what we need:

$D = E = 0$

$A + B + C = 1$

$5A + 4B + 3C = 0$

$20A + 12B + 6C = 0$

Solve for $A, B, C = 6, -15, 10$

$\Rightarrow p(x)$ is $6x^5 - 15x^4 + 10x^3$

5. (a) $\gamma(t) = (t, 1/t, 0)$

$\gamma'(t) = (1, -1/t^2, 0)$ $\quad |\gamma'(t)| = \sqrt{1 + \frac{1}{t^4}} = \frac{\sqrt{t^4+1}}{t^2}$

$\gamma''(t) = (0, 2/t^3, 0)$

$\gamma' \times \gamma'' = (0, 0, 2/t^3)$.

$$\kappa = \frac{2/t^3}{(t^4+1)^{3/2}/t^6} = \frac{2t^3}{(t^4+1)^{3/2}} = \frac{2}{(t^2+t^{-2})^{3/2}}$$

max κ = place where $t^2 + t^{-2}$ minimum, which is $t = 1$. $\kappa \to 0$ as $t \to \begin{cases}\infty \\ 0\end{cases}$

(b) $\gamma(t) = (t, \cosh t, 0)$

$\gamma'(t) = (1, \sinh t, 0)$ $\quad |\gamma'(t)| = \sqrt{1 + \sinh^2 t} = \cosh t$

$\gamma''(t) = (0, \cosh t, 0)$

$$\kappa = \frac{\cosh t}{\cosh^3 t} = \frac{1}{\cosh^2 t}$$

max κ = min for $\cosh^2 t$, so $\kappa(0) = 1$

$\kappa \to 0$ as $t \to \pm\infty$

I, 5

1. $\gamma(s) = \left(\frac{4}{5}\cos s,\ 1-\sin s,\ -\frac{3}{5}\cos s\right)$

$\gamma'(s) = T(s) = \left(-\frac{4}{5}\sin s,\ -\cos s,\ \frac{3}{5}\sin s\right)$

$T'(s) = \left(-\frac{4}{5}\cos s,\ \sin s,\ \frac{3}{5}\cos s\right)$

$|T'| = 1 \Rightarrow N = T'$ and $\kappa = 1$.

$$B(s) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{4}{5}\sin s & -\cos s & \frac{3}{5}\sin s \\ -\frac{4}{5}\cos s & \sin s & \frac{3}{5}\cos s \end{vmatrix} =$$

$\left(-\frac{3}{5},\ 0,\ -\frac{4}{5}\right)$ constant!

So γ is contained in a plane; its radius is 1 and the center is $(0,+1,0)$.

$\left(\gamma(s)-(0,1,0) = \left(\frac{4}{5}\cos s,\ -\sin s,\ -\frac{3}{5}\cos s\right)\right.$, which has length 1).

5. $A\times T = \kappa B\times T = \kappa N$ $\quad \boxed{A = \tau T + \kappa B}$

$A\times N = (\tau T + \kappa B)\times N = \tau B - \kappa T$

$A\times B = \tau T\times B = -\tau N$

1. See previous problem. The vector's length is $\sqrt{\tau^2+4\kappa^2}$.

3. $A = \mathrm{diag}(d_1, \ldots, d_n) \Rightarrow$
(a) $A^k = \mathrm{diag}(d_1^k, \ldots, d_n^k) \Rightarrow$
$\exp A = \mathrm{diag}(e^{d_1}, \ldots, e^{d_n})$

(b) $\mathrm{trace}\, A = \sum_i d_i$
$\det \exp A = \prod e^{d_i} = e^{\sum d_i} = e^{\mathrm{trace}\, A}$.

4. (a) $\mathrm{trace}\, AB =$
$\sum_i (AB)_{ii} = \sum_{ij} a_{ij} b_{ji} = \sum_j (BA)_{jj}$
$= \mathrm{trace}\, BA$

$\mathrm{trace}\,(C_1 + kC_2) = \mathrm{trace}\, C_1 + k\, \mathrm{trace}\, C_2$.

So $\mathrm{trace}\,[A,B] = \mathrm{trace}\,(AB - BA) =$
$\mathrm{trace}\, AB - \mathrm{trace}\, BA = 0$.

(b) Let ${}^TC$ = transpose of matrix C.
Given ${}^TA = -A$, ${}^TB = -B$, we have

${}^T[A,B] = {}^T(AB - BA) =$
${}^TB\,{}^TA - {}^TA\,{}^TB = [(-B)(-A)] - [(-A)(-B)] =$
$BA - AB = -[A,B]$.

(c) $$C_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad C_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$[C_{12}, C_{23}] = C_{12}C_{23} - C_{23}C_{12} =$$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = C_{13}.$$

(d) Compute all triple brackets and add them up!

$$[A,[B,C]] = A(BC-CB) - (BC-CB)A$$
$$[B,[C,A]] = B(CA-AC) - (CA-AC)B$$
$$[C,[A,B]] = C(AB-BA) - (AB-BA)C$$

Add up the right hand sides:

$$\begin{aligned} & ABC - ACB - BCA + CBA \\ &+ BCA - BAC - CAB + ACB \\ &+ CAB - CBA - ABC + BAC \\ &= 0. \end{aligned}$$

(1,1) cancels (3,3)
(1,2) cancels (2,4)
(1,3) cancels (2,1)
(1,4) cancels (3,2)
(2,2) cancels (3,4)
(2,3) cancels (3,1)

Replacement for I.3.0

Find the arc length of the exponential spiral $\gamma(t) = (e^t \cos t, e^t \sin t)$, $0 \le t \le 2\pi$.

Solution $\gamma'(t) = (e^t(\cos t - \sin t), e^t(\sin t + \cos t))$

$$|\gamma'(t)| = e^t \sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2} =$$

$$e^t \sqrt{1 - 2\sin t \cos t + 1 + 2\sin t \cos t} =$$

$$\sqrt{2} \cdot e^t.$$

$$\text{Length} = \int_0^{2\pi} \sqrt{2}\, e^t \, dt = \sqrt{2}\, e^t \Big|_0^{2\pi} =$$

$$\sqrt{2}\,(e^{2\pi} - 1).$$