Solutions to Exercises in

Unit II

(Only selected exercises from §II. 3)

SEVEN

1. Let $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the function

$$F(x,y,z) = (xy + 2z - 3xz,\ xyz + x - y - 1).$$

Then $F(1,1,1) = (0,0)$. To show that one can locally solve for any two variables in terms of the third, by the Implicit Function Theorem it is enough to show that each of

$$\frac{\partial(p,q)}{\partial(x,y)},\ \frac{\partial(p,q)}{\partial(y,z)},\ \frac{\partial(p,q)}{\partial(x,z)} \neq 0$$

if $x = y = z = 1$.

where
$$p(x,y,z) = xy + 2z - 3xz$$
$$q(x,y,z) = xyz + x - y - 1$$

$$\left.\begin{array}{l}\text{First nonzero} \Rightarrow \text{can solve for } x \,\&\, y \text{ in terms of } z \\ \text{Second nonzero} \Rightarrow \text{can solve for } y \,\&\, z \text{ in terms of } x \\ \text{third nonzero} \Rightarrow \text{can solve for } x \,\&\, z \text{ in terms of } y\end{array}\right\} \underline{\text{LOCALLY}}$$

$$\frac{\partial(p,q)}{\partial(x,y)} = \begin{vmatrix} y-3z & x \\ yz+1 & xz-1 \end{vmatrix} \underset{\text{at } (1,1,1)}{\overset{\text{eval}}{=}} \begin{vmatrix} -2 & 1 \\ 2 & 0 \end{vmatrix} \neq 0$$

$$\frac{\partial(p,q)}{\partial(y,z)} = \begin{vmatrix} x & 2-3x \\ xz-1 & xy \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \neq 0$$

$$\frac{\partial(p,q)}{\partial(x,z)} = \begin{vmatrix} y-3z & 2-3x \\ yz+1 & xy \end{vmatrix} = \begin{vmatrix} -2 & -1 \\ 2 & 1 \end{vmatrix} = 0$$

So the conclusion is that one can solve locally for either x or z, but not y, in terms of the other two variables. (SO PROBLEM NEEDS CORRECTION).

8. $(x,y) = F(s,t) = (s^2-s-2, 3t)$

For what values of (s,t) is

$$\frac{\partial(x,y)}{\partial(s,t)} \neq 0 ? \qquad \begin{vmatrix} 2s-1 & 0 \\ 0 & 3 \end{vmatrix} = \frac{\partial(x,y)}{\partial(s,t)}$$

This can be solved locally for (s,t) if $2s \neq 1$.
or $s \neq \frac{1}{2}$ or $x \neq \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 2 = -2\frac{1}{4}$.

9. $F(x,y,z) = xz + \sin xy + \cos xz - 1$.

Notize that $F(0,1,1) = 0$. To solve for x, y, z in terms of the other two, we need F_1, F_2, F_3 non zero when $(x,y,z) = (0,1,1)$.

But
$$\frac{\partial F}{\partial x}(0,1,1) = z + y\cos xy - z\sin xz \Big|_{(0,1,1)} = 2$$
$$\frac{\partial F}{\partial y}(0,1,1) = \cancel{y}\, x\cos xy \Big|_{(0,1,1)} = 0$$
$$\frac{\partial F}{\partial z}(0,1,1) = x - x\sin xz \Big|_{(0,1,1)} = 0$$

~~Hence one can solve for y and z locally, but not x.~~

Hence one can solve for x locally, but not for y and z.

1.0. $\boxed{\text{TYPO ISSUE}}$ The problem should read: Show that the system

$$e^x + e^{2y} + e^{3u} + e^{4v} = 4$$
$$e^x + e^y + e^u + e^v = 4$$

NOT ZERO

has a unique solution for u, v in terms of x, y near $(0,0,0,0)$.

We only need to show that if

$$\left.\begin{aligned} p &= e^x + e^{2y} + e^{3u} + e^{4v} - 4 \\ q &= e^x + e^y + e^u + e^v - 4 \end{aligned}\right\} \text{ then}$$

$$\left.\frac{\partial(p,q)}{\partial(u,v)}\right|_{u=v=x=y=0} \neq 0.$$

But the Jacobian is $\begin{vmatrix} 3e^{3u} & 4e^{4v} \\ e^u & e^v \end{vmatrix}$ and

if $x=y=u=v=0$ this becomes $\begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} \neq 0.$

3. We need to show F is 1-1 onto and $DF(x,y)$ is invertible for all x, y.

f is 1-1 Say $F(x,y) = F(u,v)$, so that

$$\left.\begin{aligned} xe^y + y &= ue^v + v \\ xe^y - y &= ue^v - v \end{aligned}\right| \begin{aligned} &\text{add 1st to 2nd, get } 2xe^y = 2ue^v \\ &\text{subtr 2nd from 1st, get } 2y = 2v. \end{aligned}$$

Combining the two eqns on the right we get $y = v$ & $xe^y = ue^v$. Substitute the former into the latter, getting $xe^y = ue^y$. Divide by e^y, getting $x = u$. Hence $F(x,y) = F(u,v)$ $\Rightarrow u = v$.

f is onto Solve $\begin{aligned} p &= xe^y + y \\ q &= xe^y - y \end{aligned}$ for x & y.

$p - q = 2y$, so $y = \frac{1}{2}(p-q)$. Hence

$p = xe^{\frac{1}{2}(p-q)} + \frac{1}{2}(p-q)$, so that

$x = e^{\frac{1}{2}(q-p)} \cdot \left(\frac{q-p}{2}\right)$. Since p & q were arbitrary, this shows F is onto.

Finally, check the Jacobian.

$$\frac{\partial(p,q)}{\partial(x,y)} \quad \text{where} \quad \begin{aligned} p &= xe^y + y \\ q &= xe^y - y. \end{aligned}$$

We get
$$\begin{vmatrix} e^y & xe^y + 1 \\ e^y & xe^y - 1 \end{vmatrix} =$$

$$e^y\big((xe^y - 1) - (xe^y + 1)\big) = -2e^y.$$

This is always nonzero since e^y is always positive. Hence the inverse function G, which exists since F is 1-1 and onto, is locally C^∞ near each point by the Inverse Function Thm. But this means G is also C^∞ globally, for globally $C^\infty \Leftrightarrow$ locally C^∞ near each pt.