Solutions to Exercises in

Unit III

(starred and other items omitted)

Part A — Sections III.1 – III.4

§1

O'N 4.

(b) When is this a regular parametrization?

$\sigma_u = (2u, 3u^2, 0)$ $\quad$ $\sigma_u \times \sigma_v \neq 0 \iff$

$\sigma_v = (0, 0, 1)$ $\quad$ vectors linearly indep $\iff u \neq 0$.

So σ is a regular parametrization on the region defined by $u \neq 0$. Is it 1-1?

$$\sigma(u, v) = \sigma(u', v') \iff \begin{array}{l} v = v' \\ u^2 = (u')^2 \\ u^3 = (u')^3 \end{array} \iff u = u').$$

So the answer is yes on the set where $u \neq 0$.

(c) Same approach. Start with

$\sigma_u = (1, 0, 0)$ always $\neq 0$ $\quad$ $\sigma_v = (0, 2v, 3v^2+1)$ always $\neq 0$ since 3rd coord positive.

σ_u & σ_v are always linearly indep. because $3v^2+1>0 \Rightarrow \sigma_v$ cannot be a scalar multiple of σ_u. So the parametrization is regular for all u, v. Is it 1-1?

$$\sigma(u,v) = \sigma(s,t) \Rightarrow \left.\begin{array}{l} u = s \quad \text{OK} \\ v^2 = t^2 \\ v^3+v = t^3+t \end{array}\right] \begin{array}{l}\text{does this}\\ \Rightarrow v=t?\end{array}$$

$v^2 = t^2 \Rightarrow v^2 - t^2 = 0 = (v-t)(v+t)$, so $v = \pm t$.

Is $(-t)^3 + (-t) = t^3+t$? The left side is just $-(t^3+t)$, so the only way this can happen is if $t^3+t=0$, or $t=-1$. Similar considerations show that v must be -1. But then $v = t$. Hence σ is 1-1.

O'N 5. (a) Show that $\nabla F(x,y,z) \neq 0$ if $F(x,y,z)=0$. $F(x,y,z) = (x^2+y^2)^2 + 3z^2 - 1$

When is $\nabla F(x,y,z) = 0$?

$$\nabla F(x,y,z) = \left(2(x^2+y^2)\cdot 2x,\ 2(x^2+y^2)\cdot 2y,\ 3z \right).$$

This is zero $\Longleftrightarrow$ $z=0$ and $\left.\begin{matrix}(x^2+y^2)x \\ (x^2+y^2)y\end{matrix}\right\} = 0$.

So if it's zero we have $(x^2+y^2)\cdot x^2 = (x^2+y^2)\cdot y^2 = 0$ and hence $(x^2+y^2)^2 = 0$. But this implies $x^2+y^2=0$ and hence $x=y=0$. Since $(0,0,0)$ is not a solution to $F(x,y,z)=0$, the equation defines a regular geometric surface.

(c) Here $F(x,y,z) = z(z-2)+xy-c$ and $\nabla F(x,y,z) = (y, x, 2z-2)$. The latter is $(0,0,0) \Longleftrightarrow (x,y,z) = (0,0,1)$. But if this is true, then $0 = F(0,0,1) = (-1) - c$, so that $c = +1$. Hence we get a regular surface if $c \neq 1$.

O'N 9

Show σ is regular

$\sigma_u = (1, 1, v)$ $\quad$ $\sigma_u \times \sigma_v = (?, ?, -2)$, so

$\sigma_v = (1, -1, u)$ $\quad$ the parametrization is regular.

Show σ is 1-1

$\sigma(u,v) = \sigma(s,t) \Leftrightarrow$ $\begin{aligned} u+v &= s+t \\ u-v &= s-t \\ uv &= st \end{aligned}$

Add 1st to second, getting $2u = 2s$, so that $u = s$.
But then $u+v = s+t$ & $u = s \Rightarrow v = t$. So the map is 1-1.

Show that points in image of σ satisfy eqn

$$\frac{1}{4}\left((u+v)^2 - (u-v)^2\right) = \frac{1}{4}(4uv) = uv.$$

Show that each point in graph lies in the image of σ

First solve $\begin{aligned} x &= u+v \\ y &= u-v \end{aligned}$ for u & v: $\quad u = \frac{x+y}{2}$, $\quad v = \frac{x-y}{2}$.

For these values of u and v, does $z = uv$ if $z = \frac{x^2-y^2}{4}$? Yes — do the algebra.

Additional

1 (a) distance to $(0,0,4)=$

$\sqrt{x^2+y^2+(z-4)^2}$

distance to xy plane $=|z|$. So we want

$x^2+y^2+(z-4)^2=z^2$, or

$x^2+y^2=8z-16.$

(b) distance^2 to point is $x^2+(y-2)^2+z^2$

distance^2 to plane is $(y+2)^2$. Equation is

$x^2+(y-2)^2+z^2=(y+2)^2$

$x^2+y^2=8z.$

(c) equate squares of distances

$x^2+y^2+z^2=(x-1)^2+y^2+z^2$, simplify:

$2x-1=0$ or $x=\frac{1}{2}.$

(d) $\sqrt{(x-1)^2+y^2+z^2}+\sqrt{(x+1)^2+y^2+z^2}=5.$

$\boxed{5}$

2 Write out the equations

$$|x|^2 - 2a\cdot x + |a|^2 = |x|^2 - 2b\cdot x + |b|^2 =$$

$$|x|^2 - 2c\cdot x + |c|^2 = |x|^2.$$

Subtract $|x|^2$ from each one. We get

$$0 = -2a\cdot x + |a|^2 = 2b\cdot x + |b|^2 = 2c\cdot x + |c|^2$$

or
$$\left.\begin{aligned} a\cdot x &= \frac{|a|^2}{2} \\ b\cdot x &= \frac{|b^2|}{2} \\ c\cdot x &= \frac{|c^2|}{2} \end{aligned}\right\}$$ 3 linear equns in $x = u, v, w$.

The matrix with rows $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is invertible because the vectors are lin indep and hence we can solve uniquely for x.

~~2~~4. (Easier) $x = u \Rightarrow$
$y = x^2 + v$
and $v^2 = (y - x^2)^2$, so

the image of the surface consists of points whose coordinates satisfy $z^2 = (x - y^2)^2$.
[But not all such points — notice that $z \geq 0$ in the parametrization by $z^2 - (x-y^2)^2$ has solutions with $z < 0$.].

3. $x = (u^2 - v^2) = (u - v)(u + v) =$
$y \cdot (y + 2v) = y^2 + 2yv.$

Solve for v $\quad v = \dfrac{x - y^2}{2y}$. Call this $\dfrac{Q(x,y)}{2y}$.

But now $y = u - v \Rightarrow u = y + v$; call this $\dfrac{P(x,y)}{2y}$.
$= y + \dfrac{x - y^2}{2y} = \dfrac{2y^2 + x - y^2}{2y}.$

We then get $z =$
$\left(\dfrac{P(x,y)}{2y}\right)^3 + 3\left(\dfrac{Q(x,y)}{2y}\right)$ or equivalently

$8y^3 z = P(x,y)^3 + 12y^2 Q(x,y).$

§2

0. For the graph of $y = f(x)$ the parametric equations are
$x = t \quad y = f(t)\cos u \quad z = f(t)\sin u$, so in this case we have $\gamma(t) = (t, e^{-t}\cos u, e^{-t}\sin u)$

1. (i) Critical points are those for which

$$\nabla (x+y+z-1)^2 = 0.$$

The gradient is $2(x+y+z-1)\cdot\nabla(x+y+z-1)$
$= 2(x+y+z-1)\cdot(1,1,1)$ and the critical points are those for which $x+y+z-1=0$.
The critical value is zero.

(ii) So the level surfaces are those for which

$$(x+y+z-1)^2 = c^2 \quad \text{for } c > 0.$$

(iii) Critical points = those where $\nabla xyz^2 = 0$

But $\nabla(xyz^2) = (yz^2, xz^2, 2xyz)$

When is this equal to zero?

3rd coord at least one of x, y, z is zero.
2nd one of x, z is zero
3rd one of y, z is zero.

Let W = set where $w = 0$, $w = x, y$ or z.
Then the critical points are $(X \cup Z) \cap (Y \cup Z)$
$= (X \cap Y) \cup Z$, so of one of the forms $(0, 0, z)$ or $(x, y, 0)$
In all cases the value of the function is zero.
Hence the level sets $xyz^2 = c$ are geometric regular surfaces provided $c \neq 0$.

2. If Σ is a geometric regular surface, then for each $p \in \Sigma$ there is a change of variables $h : (-u, u) \times (-u, u) \times (-u, u) \longleftrightarrow V$, where V is a region in $\mathbb{R}^3$ containing p, and under h the set $\Sigma \cap h[V]$ corresponds to $(-u, u) \times (-u, u) \times \{0\}$. Let g be as given, and let $x \in g[\Sigma]$. If $x = g(p)$ and h is as above, then the composite $g \circ h$ has all the right properties.

3. TYPO $\sigma(\theta, z) = (f(\theta)\cos\theta, f(\theta)\sin\theta, z)$.

Regular parametrization?

$\sigma_\theta = (f'(\theta)\cos\theta - f(\theta)\sin\theta,\ f'(\theta)\sin\theta + f(\theta)\cos\theta,\ 0)$

$\sigma_z = (0, 0, 1)$. Need to show $\sigma_\theta \neq 0$;

if so, then $\sigma_\theta \times \sigma_z \neq 0$.

But $\sigma_\theta = 0 \Rightarrow f'(\theta)\cos\theta = f(\theta)\sin\theta$

$-f'(\theta)\sin\theta = f(\theta)\cos\theta$.

Now $f(\theta) > 0$, and if we square both equations and add them, then we get

$f(\theta)^2 = f(\theta)^2(\cos^2\theta + \sin^2\theta) = f'(\theta)^2(\cos^2\theta + \sin^2\theta) = f'(\theta)^2$,

so that $f'(\theta) = \pm f(\theta)$ and hence

$$(\cos\theta, \sin\theta) = \pm(-\sin\theta, \cos\theta) =$$

$$(\cos(\theta + \tfrac{\pi}{2} + k\pi), \sin(\theta + \tfrac{\pi}{2} + k\pi)),$$

where $k = 0, 1$.

Now $(\cos\theta, \sin\theta) \neq$

$(\cos(\theta + \alpha), \sin(\theta + \alpha))$ for $0 < \alpha < 2\pi$

so the derived eqns are impossible.

Hence the parametrization is regular.

Finally, the normal vector is

$$\sigma_\theta \times \sigma_z =$$

$$(f'(\theta)\sin\theta + f(\theta)\cos\theta,\ f(\theta)\sin\theta - f'(\theta)\cos\theta,\ 0)$$

4. OMIT

5. Computing $\sigma_u \times \sigma_v$ is messy, so we shall prove that the points of the surface are those for which $x^3+y^3=z^2$, and assuming this we shall prove that we have a regular surface.

Regularity: Surface given by

F: $x^3+y^3-z^2=0$

$$\nabla F(x,y,z) = (3x^2, 3y^2, -2z).$$

At least one of x, y, z is non zero, so at least one coord of $\nabla F(x,y,z)$ is nonzero, proving regularity.

Now we can compute

$$x^3+y^3 = \frac{u^6}{(1+v^3)^3} + \frac{u^6v^3}{(1+v^3)^3} =$$

$$\frac{u^6(1+v^3)}{(1+v^3)^3} = \frac{u^6}{(1+v^3)^2} = z^2.$$

So the image of the parametrization lies in the set $x^3+y^3=z^2$.

Next, show σ is 1-1. Suppose that

$$\frac{u^2}{1+v^3} = \frac{a^2}{1+b^3}, \quad \frac{u^2v}{1+v^3} = \frac{a^2b}{1+b^3}, \text{ and}$$

$$\frac{u^3}{1+v^3} = \frac{a^3}{1+b^3}.$$

Divide 3rd by 1st, get $u=a$.
Divide 2nd by 1st, get $v=b$.

Now show every point of $x^3+y^3=z^2$ comes from parametrization. In other words we need to solve

$$x = \frac{u^2}{1+v^3}, \quad y = \frac{u^2 v}{1+v^3}, \quad z = \frac{u^3}{1+v^3}$$

for u and v. $\frac{z}{x} = u$, $\frac{y}{x} = v$.

Finally, we need to show if we substitute the expressions in x, y, z for $u + v$, these expressions simplify to x, y, z.

For example, $\frac{u^2}{1+v^3} = \frac{\frac{z^2}{x^2}}{1+\frac{y^3}{x^3}} =$

$$\frac{xz^2}{x^3+y^3} = \frac{x(x^3+y^3)}{x^3+y^3} = x.$$

To finish, $z = xu = \frac{u^3}{1+v^3}$, $y = xv = \frac{u^2 v}{1+v^3}$.

§3

O'N 10 (a) The normal vector is $\nabla(x^2+y^2+(z-1)^2-1) = (2x, 2y, 2z-2)$

Evaluate at $(0,0,0)$ $\rightsquigarrow$ get $(0,0,-2)$.

So the tangent plane has eqn.

$(0, 0, -2) \cdot (x, y, z) = 0$

since point of tangency is $(0,0,0)$.

(b) <u>TYPO</u> $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{18} = 1.$

$\nabla\left(\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{18} - 1\right) = \left(\frac{x}{2}, \frac{y}{8}, \frac{z}{9}\right)$

Evaluation at $(1, -2, 3)$ yields $\left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{3}\right)$ and the tangent plane equation becomes

$(6, -3, 4) \cdot (x-1, y+2, y-3) = 0.$

(c) $\sigma(2, \frac{\pi}{4}) = (\sqrt{2}, \sqrt{2}, \frac{\pi}{2})$

$\sigma_u = (\cos v, \sin v, 0)$

$\sigma_v = (-u \sin v, u \cos v, 1).$

$\sigma_u \times \sigma_v = (\sin v, -\cos v, u) \neq 0$

at $(2, \frac{\pi}{4})$ this vector is

$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 2\right) \sim (\sqrt{2}, -\sqrt{2}, 4).$

The tangent plane eqn is then

$$(\sqrt{2}, -\sqrt{2}, 4)\cdot(x-\sqrt{2}, y-\sqrt{2}, z-\tfrac{\pi}{2}).$$

Additional exercises

0. Normal vector is $\overset{z=f(x,y)}{(-f_u, -f_v, 1)}$.

(a) $f(u,v)=\sqrt{uv} \Rightarrow$

$$\frac{\partial f}{\partial u}=\frac{1}{2}\sqrt{\frac{v}{u}}, \quad \frac{\partial f}{\partial v}=\frac{1}{2}\sqrt{\frac{u}{v}}.$$

at $(1,1,1)$ the normal vector is

$$\left(-\tfrac{1}{2}, -\tfrac{1}{2}, 1\right) \sim (-1,-1,2)$$

$\therefore$ Tangent plane equation is

$$(-1,-1,2)\cdot(x-1, y-1, z-1)=0.$$

(b). First solve $(2u\cos v, 3u\sin v, u^2)=$ $(0,6,4)$ for u & v:

$u=\pm 2 \qquad 2u\cos v=0$

$\Rightarrow v=\pm\frac{\pi}{2}.$

Since $3u \sin v = 6$, we have either

$(u, v) = (2, \frac{\pi}{2})$ or $(-2, -\frac{\pi}{2})$.

Find $\underline{X}_u \times \underline{X}_v$.

$$\begin{array}{l|l} & \text{eval at } (\pm 2, \pm\frac{\pi}{2}) \\ \underline{X}_u = (2\cos v,\ 3\sin v,\ 2u) & (0, \pm 3, \pm 4) \\ \underline{X}_v = (-2u\sin v,\ 3u\cos v,\ 0) & (\mp 2, 0, 0) \end{array}$$

The normal vector is (for both (u,v) choices) $(0,3,4) \times (2,0,0) = (0,+8,-6)$.

Tangent plane equation is

$$(0, 8, -6) \cdot (x, y-6, z-4) = 0.$$

(c) OMIT

1. Cylinder parametrization:

$(x(t), y(t), w)$

Normal direction

$(x'(t), y'(t), 0) \times (0, 0, 1) =$

$(y'(t), -x'(t), 0)$.

Hold t_0 constant to get the vertical ruling. Let's say $t = C$

Tangent plane at $(x(C), y(C), B)$ is
$\overset{''}{P}, \overset{''}{Q}, \overset{''}{B}$

$$(y'(C), -x'(C), 0) \cdot (u - P, v - Q, z - B) = 0.$$

$$= y'(C)(u-P) + x'(C)(v-Q).$$

But this equation does not depend upon the third coordinate B.

4. (unstarred) pt. $= (a, b, 0)$

$$\nabla(x^2+y^2+z^2-1) = (2x, 2y, 2z) =$$
$$(2a, 2b, 0)$$

So the normal direction is always perpendicular to $(2a, 2b, 0) \leftarrow$ note that $a^2+b^2=1$ $\Rightarrow$ this vector is nonzero.

Tangent plane equation is

$$(a, b, 0) \cdot (x-a, y-b, z) = 0 =$$
$$ax + by - (a^2+b^2).$$

We need to show that this line is disjoint from the z-axis $(0,0,t)$.
Show there is no t such that

$$\underset{0}{ax} + \underset{0}{by} = a^2+b^2 \rightarrow \text{positive since } (a,b)\neq(0,0)$$

So if (x,y) is on the tangent plane then $ax+by>0$. But a point on the z-axis has $x=y=0$, so no such point lies on the plane.

5. $z = xf\left(\frac{y}{x}\right)$

$$\frac{\partial z}{\partial x} = f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) =$$

$$f\left(\frac{y}{x}\right) - \frac{y}{x} f'\left(\frac{y}{x}\right).$$

$$\frac{\partial z}{\partial y} = xf'\left(\frac{y}{x}\right)\cdot\frac{1}{x} = f'\left(\frac{y}{x}\right)$$

~~So the tangent plane at $\left(a, b, af\left(\frac{b}{a}\right)\right)$ is~~

$$\cancel{z - af\left(\frac{b}{a}\right) = \left[f\left(\frac{b}{a}\right) - \frac{b}{a}f'\left(\frac{b}{a}\right)\right](x-a) + \frac{1}{a}f'\left(\frac{b}{a}\right)(y-b)}$$

To prove that O lies in the tangent plane, we need to show the tangent plane's equation has the form $z = Px + Qy$, with no constant terms. Let $(a, b, af(\frac{b}{a}))$ be a point on the surface, so the tangent plane has equation

$$z - af\left(\frac{b}{a}\right) = \left[f\left(\frac{b}{a}\right) - \frac{b}{a}f'\left(\frac{b}{a}\right)\right](x-a) + f'\left(\frac{b}{a}\right)(y-b) =$$

$$\boxed{???}\,x + \boxed{????}\,y - af\left(\frac{b}{a}\right) + \cancel{bf'\left(\frac{b}{a}\right)} - \cancel{bf'\left(\frac{b}{a}\right)} =$$

CANCEL

$$\boxed{???}\,x + \boxed{????}\,y - af\left(\frac{b}{a}\right),$$ and

this reduces to

$$z = \boxed{???}\,x + \boxed{????}\,y.$$ So $(0,0,0)$ is always in the tangent plane.

7. Find the common points:

$$|x|^2 = 1, \quad |x-a|^2 = 1.$$
$$\qquad\qquad \| $$
$$|x|^2 - 2a\cdot x + |a|^2$$

Subtract 1st from 2nd: $-2a\cdot x + |a|^2 = 0$ defines common pts.

$\Longrightarrow$ lie on plane $2a\cdot x = |a|^2$.

For these points the normal direction to the spheres are x and $x-a$, and we want to know when $x \perp x-a$; i.e., $|x|^2 = a\cdot x$, when $|x|^2 = 1$ & $|x-a|^2 = 1$.

Now $\underbrace{|x|^2}_{(=1)} - 2a\cdot x + |a|^2 = 1$ | hence $a\cdot x = |x|^2 = 1$

$$1 \quad -2 \quad + |a|^2 = 1.$$

Hence $x \perp (x-a)$ means $|a|^2 = 2$.

Conversely, $|a|^2 = 2 \Longrightarrow a\cdot x = 1 \Rightarrow x \perp x-a$.

<u>Generalization</u> $|a|^2 = r^2 + s^2$.

8(a) $f(ax, ay, az) =$

$p a^d x^d + q a^d y^d + r a^d z^d =$

$a^d (p x^d + q y^d + r z^d) = a^d f(x, y, z).$

proves 1st part.

For the 2nd part,

$\nabla f(x, y, z) = (p d x^{d-1}, q d y^{d-1}, r d z^{d-1})$

Now atleast one of x, y, z is non zero, and $p, q, r, d \neq 0$ (d odd). Since d is odd, atleast one of $x^{d-1}, y^{d-1}, z^{d-1}$ is non zero $\Rightarrow$ same for $\left.\begin{matrix} p d x^{d-1} \\ q d y^{d-1} \\ r d z^{d-1} \end{matrix}\right\}$ Hence $\nabla f \neq 0$ if $(x, y, z) \neq (0, 0, 0)$.

Since $c > 0$ and $f(x, y, z) = c \Rightarrow$

$(x, y, z) \neq (0, 0, 0)$ [otherwise $f(0, 0, 0) = 0$]

$\nabla f \neq 0$ when $f = c$, so we get a regular smooth surface.

(b) OMIT.