Solutions to Exercises in

Unit III

Part B – Sections III.4–6

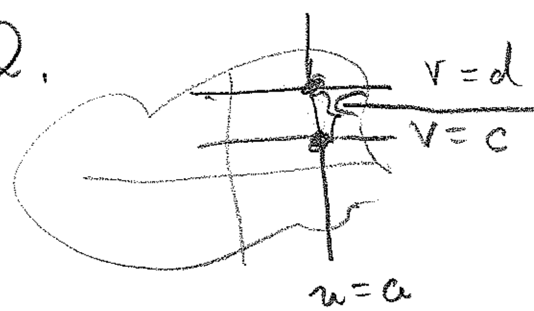
§4 (omit 7)

1. $X_u = (f'(u)\cos v, f'(u)\sin v, 0)$

$X_v = (-f(u)\sin v, f(u)\cos v, g'(v))$

$X_u \cdot X_u = f'(u)^2 \qquad X_u \cdot X_v = 0$

$X_v \cdot X_v = f(u)^2 + g'(v)^2.$

2. 

These are the curves in the problem

$u(t) = a$
$v(t) = t \qquad c \le t \le d$

CORRECTION: Suppose the FFF is

$1\,du\,du + f(v)\,dv\,dv$. (Hence $f(v) > 0$)

Then length $= \int_c^d \sqrt{1\cdot\left(\frac{du}{dt}\right)^2 + f(t)^2\left(\frac{dv}{dt}\right)^2}\, dt =$

$\int_c^d \sqrt{f(t)}\, dt$, which does not depend on the choice of a.

3. In each case, find the partial derivatives + take dot products.

(i) $X_u = (a\cos u\cos v, b\cos u\sin v, -c\sin u)$
$X_v = (-a\sin u\sin v, -b\sin u\cos v, 0)$.

$E = X_u\cdot X_u = a^2\cos^2 u\cos^2 v + b^2\cos^2 u\sin^2 v + c^2\sin^2 u$
$F = X_u\cdot X_u = -(a^2+b^2)\cos u\cos v\sin u\sin v$.
$G = X_v\cdot X_v \quad (a^2+b^2)\sin^2 u$. (regular everywhere $u \neq k\pi$)

(ii) $X_u = (a\cos v, a\sin v, 2u)$
$X_v = (-au\sin v, au\cos v, 0)$] regular everywhere with $u \neq 0$

$E = X_u\cdot X_u = a^2+4u^2 \quad F = X_u\cdot X_v = 0$
$G = X_v\cdot X_v = a^2u^2$.

(iii) $X_u = (a\cosh v,\ a\sinh v,\ 2u)$

$X_v = (au\sinh v,\ au\cosh v,\ 0)$

$E = X_u \cdot X_u = a^2(\cosh^2 v + \sinh^2 v) + 4u^2$

$F = X_u \cdot X_v = 2u\,a^2 \cosh v \sinh v.$

$G = X_v \cdot X_v = a^2u^2(\cosh^2 v + \sinh^2 v)$

(iv) $X_u = (a\cosh u\cos v,\ b\cosh u\sin v,\ c\sinh u)$

$X_v = (-a\sinh u\sin v,\ b\sinh u\cos v,\ 0)$

$E = X_u \cdot X_u = a^2\cosh^2 u\cos^2 v + b\cosh^2 u\sin^2 v + c\sinh^2 u$

$F = X_u \cdot X_v = (b^2 - a^2)\sinh u\cosh u\sin v\cos v.$

$G = X_v \cdot X_v = a^2\sinh^2 u\sin^2 v + b^2\sinh^2 u\cos^2 v$

(v) $X_z = (\cos v, \sin v, 1)$

$X_v = (-z\sin v,\ z\cos v,\ 0)$

$E = X_z \cdot X_z = 2$

$F = X_z \cdot X_v = 0$

$G = X_v \cdot X_v = z^2.$

TYPO $\boxed{\text{switch } E \text{ and } G \;\Big|\; E=1, \; G=G(v).}$

4. Parametrize the graph by arclength $\gamma(s) = (x(s), y(s))$. The surface becomes $\sigma(u,v) = (x(u), y(u)\cos v, y(u)\sin v)$.

Then $\sigma_u \cdot \sigma_u = (x')^2 + (y')^2 = 1$

$\sigma_u \cdot \sigma_v = 0$

$\sigma_v \cdot \sigma_v = y^2.$

§5

OMIT

§6

1. $F = z - x^3 \qquad G = y - x^2$

$\nabla F = (-3x^2, 0, 1) \quad \nabla G = (-2x, 1, 0).$

Say $(x, y, z) = (a, a^2, a^3)$.

$\nabla F = (-3a^2, 0, 1) \quad \nabla G = (-2a, 1, 0)$

Find $\measuredangle\, \nabla F, \nabla G$.

The angle is the arc cosine of

$$\frac{\nabla F . \nabla G}{|\nabla F| \cdot |\nabla G|} = \frac{6a^3}{\sqrt{1+9a^4}\,\sqrt{1+4a^2}}.$$

2. $x(\theta) = (1+\cos\theta, \sin\theta, 2\sin\theta/2)$

Surfaces sphere $0 = F(x,y,z) = x^2+y^2+z^2-4$.

cyl. $0 = G(x,y,z) = (x-1)^2+y^2-1$,

$\nabla F = (2x, 2y, 2z)$

$\nabla G = (2(x-1), 2y, 0)$.

<u>Curve is regular:</u>

$x'(\theta) = (1-\sin\theta, \cos\theta, \cos\theta/2)$.

If $x'(\theta) = (0,0,0)$ then $\left.\begin{array}{l}\sin\theta = 1\\ \cos\theta = 0\end{array}\right] \Rightarrow \theta = \frac{\pi}{2}$

$\cos(\theta/2) = 0$. But $\cos\frac{\pi}{4} \neq 0$. so $x'(\theta)$ never zero

<u>Compute grad vects at $x(\theta)$</u>

$\nabla F = 2x(\theta)$

If $\nabla G + \nabla F$ are linearly dependent, and $y \neq 0$, then 2nd coords of ∇G, ∇F equal & nonzero $\Rightarrow \nabla F = \nabla G$. Hence

$2z = 0$ & $2(x-1) = 2x$.

But $2(x-1) = 2x \Rightarrow 0 = 2$, impossible. So ∇F & ∇G are always lin indep. if $y \neq 0$.

3. Find the normal vectors at (a, b, c)

$$\left.\begin{array}{ll} z = f(x,y) & (-f_x, -f_y, 1) = N(f) \\ z = g(x,y) & (-g_x, -g_y, 1) = N(g) \end{array}\right\} \text{normals to surfaces}$$

Surfaces are transverse means $N(f) \times N(g) \neq 0$ ($\Leftrightarrow N(f), N(g)$ lin. indep.).

But $N(f) \times N(g) =$

$$\left(\sim, \sim, \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} \right)$$

not important exactly what these are!!

and $\nabla f \times \nabla g = \left(0, 0, \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix}\right)$.

Hence $\nabla f \times \nabla g \neq 0 \Rightarrow$ third coord. of $N(f) \times N(g)$ nonzero & hence $N(f) \times N(g) \neq 0$.