Solutions to Exercises in

Unit IV

Part A — Sections IV.2-3

Q'N 3 CLAIM The rank of the shape operator equals the rank of $\begin{pmatrix} e & f \\ f & g \end{pmatrix}$ where e, f, g are the coefficients of the SFF. As in §IV.3 of the notes, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gives the shape operator, then

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

↑ invertible since $EG - F^2 > 0$.

Hence the ranks of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} e & f \\ f & g \end{pmatrix}$ are $=$;

for ① $\det\begin{pmatrix} e & f \\ f & g \end{pmatrix} \Big/ \det\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ So both zero or both nonzero.

② rank 2 $\quad \det \neq 0$.
rank 1 $\quad \det = 0$ but matrix $\neq 0$
rank 0 $\quad$ matrix is zero

So we need only compute the rank of $\begin{pmatrix} e & f \\ f & g \end{pmatrix}$.

(c) Let $\sigma(u, v, 2u^2+v^2)$. Then a normal is given by $\Omega = (4u, 2v, -1)$ NOT A UNIT VECTOR.

and

$$e = \frac{1}{|\Omega|}(\Omega \cdot \sigma_{uu}) = \frac{1}{|\Omega|}(4u, 2v, -1)\cdot(0, 0, 4) = \frac{-4}{|\Omega|}$$

$$f = \frac{1}{|\Omega|}(\Omega \cdot \sigma_{uv}) = \frac{1}{|\Omega|}(4u, 2v, -1)\cdot(0, 0, 0) = 0$$

$$g = \frac{1}{|\Omega|}(\Omega \cdot \sigma_{vv}) = \frac{1}{|\Omega|}(4u, 2v, -1)\cdot(0, 0, 2) = \frac{-2}{|\Omega|}.$$

This means $\begin{pmatrix} e & f \\ f & g \end{pmatrix}$ = NONZERO $\cdot \begin{pmatrix} -4 & 0 \\ 0 & -2 \end{pmatrix}$, so rank$=2$.

(d) Now take $\sigma(u, v) = (u, v, uv^2)$.

Then $\Omega = (v^2, 2uv, -1)$.

Now
$$\left.\begin{aligned}\sigma_{uu} &= (0,0,0)\\ \sigma_{uv} &= (0,0,2v)\\ \sigma_{vv} &= (0,0,2u)\end{aligned}\right\} \Longrightarrow \quad \begin{aligned} e &= 0\\ f &= \frac{-2v}{|\Omega|}\\ g &= \frac{-2u}{|\Omega|}\end{aligned}$$

These all vanish at $u=v=w=0$, so the shape operator is zero at $(0,0,0)$.

Additional

1. If $e=f=g=0$, then as in the discussion before the previous exercise we know that the shape operator is zero, which means $\frac{\partial N}{\partial u} = \frac{\partial N}{\partial v} = 0$, so that N is locally constant. Hence if $\gamma(t)$ is a smooth curve in Σ we have $N(\gamma(t)) = N_0$ (fixed).

at least locally near t_0 s.t. $\gamma(t_0)=p$.
But this means that $N_0 \cdot \gamma'(t) \equiv 0$

because $N(\gamma(t)) \perp \gamma'(t)$ by definition of the normal vectors.

To show $\gamma(t)$ is contained in a plane, we need to show that $N_0 \cdot \gamma(t) = N_0 \cdot p$ where $p = \gamma(t_0)$; this will follow if $\frac{d}{dt} N_0 \cdot \gamma(t) = 0$, and that is true because $\frac{d}{dt}(N_0 \cdot \gamma) = N_0 \cdot \gamma'$.

§3

1. (i) Suppose A has two positive eigenvalues λ_1, λ_2. Then $\det A = \lambda_1 \lambda_2 > 0$. Also, $a_{11} = \langle Ae_1, e_1 \rangle$ where e_1 is the first unit vector. Write $e_1 = x_1 u_1 + x_2 u_2$ where u_1, u_2 are orthonormal and $Au_j = \lambda_j u_j$. Then $\langle Ae_1, e_1 \rangle \geq \min(\lambda_1, \lambda_2)$ which is positive.

by Rayleigh's Principle

Conversely, suppose $\det A$, $a_{11} > 0$. Then $\det A > 0$ $\det A = \lambda_1 \lambda_2 \Rightarrow \lambda_1$ & λ_2 are either both positive or negative. If both are negative, then as before we can use Rayleigh's Principle to see that $a_{11} \leq \max \lambda_1, \lambda_2 < 0$. Since this is false, we must have $\lambda_1, \lambda_2 > 0$

(iii) $\det A = \lambda_1 \lambda_2 = 0$ Let's say $\lambda_2 = 0$.
$\operatorname{trace} A = \lambda_1 + \lambda_2 > 0$

Then $\operatorname{trace} A = \lambda_1$, so $\lambda_1 > 0$.

(iv) In (ii), have $\det A > 0$, $a_{11} < 0 \Leftrightarrow \lambda_1, \lambda_2 < 0$. In (iii), have $\det A = 0$ & $\operatorname{tr} A < 0$ $\Leftrightarrow 0 = \lambda_2 > \lambda_1$.

(iii) $\det A = \lambda_1 \lambda_2 < 0$ means one factor is negative and the other is positive, and conversely.

2. Since A has an orthonormal basis of eigenvectors, we have $\det(A-tI) = (\lambda_3-t)(\lambda_2-t)(\lambda_1-t)$, and $\det A = \lambda_1\lambda_2\lambda_3$ by standard results of linear algebra.

(i) If $\lambda_1, \lambda_2, \lambda_3 > 0$, then $\det A = \lambda_1\lambda_2\lambda_3 > 0$.

(ii). The matrix B is 2×2 and symmetric, and $\sum_{i,j=1}^{2} b_{ij}x_ix_j = \sum_{i,j=1}^{3} a_{ij}x_ix_j$ if we set $x_3 = 0$ & leave x_1 & x_2 as is.

Say $\lambda_1 \geq \lambda_2 \geq \lambda_3$

Let $J: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear map sending $v = (x_1, x_2)$ to $(x_1, x_2, 0)$.

Then $$\frac{\langle Bv, v\rangle}{\langle v, v\rangle} = \frac{\langle AJv, Jv\rangle}{\langle Jv, Jv\rangle}$$ if v, hence also $Jv \neq 0$.

Clearly if the eigenvalues of B are $\alpha_1 \geq \alpha_2$ we have
$$\alpha_2 = \min \frac{\langle Bv, v\rangle}{\langle v, v\rangle} \geq \min \frac{\langle Ax, x\rangle}{\langle x, x\rangle} = \lambda_3 > 0$$
since $\frac{\langle Bv, v\rangle}{\langle v, v\rangle} = \frac{\langle AJv, Jv\rangle}{\langle Jv, Jv\rangle}$ and

$$\min \frac{\langle Ax, x\rangle}{\langle x, x\rangle} \le \text{minimum of the smaller value set } \frac{\langle AJv, Jv\rangle}{\langle Jv, Jv\rangle} = \frac{\langle Bv, v\rangle}{\langle v, v\rangle}.$$

Thus $\alpha_2 \ge \lambda_3 > 0 \Rightarrow \alpha_1 \ge \alpha_2 \ge \lambda_3 > 0$,

so B has two positive eigenvalues.

In particular, we must have $\det B > 0$.