Solutions to Exercises in

Unit IV

Part B – Section IV.4

O'N 3. These are really just problems about symmetric 2×2 matrices:

(a) Prove that if u_1 & u_2 are orthonormal then $\frac{1}{2}(\langle Au_1, u_1\rangle + \langle Au_2, u_2\rangle) = \frac{1}{2}\text{trace } A$

(b) $\frac{1}{2}\text{ trace } A = \frac{1}{2\pi}\int_0^{2\pi} \langle A v_\theta, v_\theta\rangle \, d\theta$

where $v_\theta = \cos\theta u_1 + \sin\theta u_2$.

Solution to (a) Let y_1 & y_2 be orthonormal eigenvectors with eigenvalues λ_1 & λ_2.

Write $u_1 = \cos\theta y_1 + \sin\theta y_2$. Then

u_2 must be $\cos(\theta \pm \frac{\pi}{2}) y_1 + \sin(\theta \pm \frac{\pi}{2}) y_2 =$

$\pm(-\sin\theta y_1 + \cos\theta y_2)$.

Hence $\langle Aw_1, w_1\rangle =$ $\lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta$ | $\langle Au_2, u_2\rangle =$ $\lambda_1 \sin^2\theta + \lambda_2 \cos^2\theta$

so $\langle Au_1, w_1\rangle + \langle Au_2, u_2\rangle = \lambda_1 + \lambda_2$.

Divide by 2 and use $\lambda_1 + \lambda_2 = \text{trace } A$ to get the formula to be proved.

(b) We might as well take u_1 & u_2 to be y_1 & y_2, for if $w_1 = \cos\alpha\, y_1 + \sin\alpha\, y_2$ then $V_\theta = \cos(\alpha+\theta)\, y_1 + \sin(\alpha+\theta)\, y_2$ and the two integrals (w.r.t. u's & y's) agree by periodicity of sin & cos plus a change of variables. But then $\langle AV_\theta, V_\theta\rangle =$

$\lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta$. & the integral is

$$\int_0^{2\pi} (\lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta)\, d\theta = \lambda_1 \int_0^{2\pi} \cos^2\theta\, d\theta + \lambda_2 \int_0^{2\pi} \sin^2\theta\, d\theta.$$

Now $\int_0^{2\pi} \cos^2\theta\, d\theta = \int_0^{2\pi} \sin^2\theta\, d\theta = \pi$, and

this shows the integral is $\pi(\lambda_1 + \lambda_2)$.

Hence $\frac{1}{2}\text{trace } \alpha = \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2\pi}$ (integral).

Additional

1 OMIT MÖBIUS STRIP, ELLIPSOID, PARABOLOIDS

GAUSSIAN CURVATURE ONLY

One sheeted hyperboloid

$x^2+y^2-z^2=1.$ $\quad F = x^2+y^2-z^2-1$

$\nabla F = (2x, 2y, -2z)$

$\sigma(u,v) = (\cosh u \cos v, \cosh u \sin v, \sinh u).$

$\sigma_u = (\sinh u \cos v, \sinh u \sin v, \cosh u)$

$\sigma_v = (-\cosh u \sin v, \cosh u \cos v, 0)$

FFF: $E = \sigma_u \cdot \sigma_u = \cosh^2 u + \sinh^2 u.$

$F = \sigma_u \cdot \sigma_v = 0$

$G = \sigma_v \cdot \sigma_v = \cosh^2 u.$

SFF:

$$\left.\begin{aligned} e &= \frac{1}{|\nabla F|} F \cdot \sigma_{uu} \\ f &= \frac{1}{|\nabla F|} F \cdot \sigma_{uv} \\ g &= \frac{1}{|\nabla F|} F \cdot \sigma_{vv} \end{aligned}\right]$$

Substitute the values as on the left. We get.

$N = \frac{1}{|\nabla F|} \cdot F$

$\nabla F = (2\cosh u \cos v, 2\cosh u \sin v, -2\sinh u)$

So

$|\nabla F| = 2\sqrt{\cosh^2 u + \sinh^2 u}$

$\sigma_{uu} = \sigma$

$\sigma_{uv} = (-\sinh u \sin v, \sinh u \cos v, 0)$

$\sigma_{vv} = (-\cosh u \cos v, -\cosh u \sin v, 0).$

$$e = \frac{2\,1(\cosh^2 u - \sinh^2 u)}{\cancel{2}(\cosh^2 u + \sinh^2 u)^{1/2}} = \frac{1}{(\cosh^2 u + \sinh^2 u)^{1/2}}$$

$$f = \frac{2(0)}{\cancel{2}(\cosh^2 u + \sinh^2 u)^{1/2}} = 0$$

$$g = \frac{-\cancel{2}\cosh^2 u}{\cancel{2}(\cosh^2 u + \sinh^2 u)^{1/2}} = \frac{-\cosh^2 u}{(\cosh^2 u + \sinh^2 u)^{1/2}}$$

$$K = \frac{eg - f^2}{EG - F^2} = \frac{\dfrac{-\cosh^2 u}{(\cosh^2 u + \sinh^2 u)^{1/2}}}{\cosh^2 u(\cosh^2 u + \sinh^2 u)} =$$

$$\frac{-1}{(\cosh^2 u + \sinh^2 u)^{3/2}}.$$ Always negative.

<u>Two sheeted hyperboloid</u>

$$x^2 - y^2 - z^2 - 1 = 0 = F$$

$$\nabla F = (2x, -2y, -2z).$$

Same approach, but now

$$\sigma(u, v) = (\cosh u, \sinh u \cos v, \sinh u \sin v).$$

$N(u,v) = \frac{1}{|\nabla F|} \nabla F$ written in terms of u, v

$= \frac{1}{\sqrt{\cosh^2 u + \sinh^2 u}} \left(\cosh u, \overline{\sinh u \cos v}, \overline{\sinh u \sin v} \right).$

$\boxed{\text{need } u \neq 0}$

$\sigma_u = (\sinh u, \cosh u \cos v, \cosh u \sin v)$

$\sigma_v = (0, -\sinh u \sin v, \sinh u \cos v)$

otherwise $\sigma_v = 0$

$\sigma_{uu} = \sigma = (\cosh u, \sinh u \cos v, \sinh u \sin v).$

$\sigma_{uv} = (0, -\cosh u \sin v, \cosh u \cos v)$

$\sigma_{vv} = (0, -\sinh u \cos v, -\sinh u \sin v).$

$E = \cosh^2 u + \sinh^2 u$

$F = 0$

$G = \sinh^2 u.$

$EG - F^2 = \sinh^2 u (\cosh^2 u + \sinh^2 u).$

$e = N \cdot \sigma_{uu} = \frac{1}{\sqrt{\cosh^2 u + \sinh^2 u}}$

$f = N \cdot \sigma_{uv} = 0$

$g = N \cdot \sigma_{vv} = \frac{\sinh^2 u}{\sqrt{\cosh^2 u + \sinh^2 u}}$

Hence

$$K = \frac{eg - f^2}{EG - F^2} = \frac{\sinh^2 u}{\sinh^2 u \,(\cosh^2 u + \sinh^2 u)^{3/2}} =$$

$$\frac{1}{(\cosh^2 u + \sinh^2 u)^{3/2}} > 0.$$

This is at least true for $u \neq 0$. One can use a limiting argument and the alternate parametrization(s) $x = \pm\sqrt{1 + y^2 + z^2}$ to conclude that $K > 0$ everywhere (look at $\lim_{u \to 0} K(u,v) = \frac{1}{1^{3/2}} = 1$).

To study the elliptic paraboloid, look at $x^2 + y^2 - z = 0$, so $F = x^2 + y^2 - z^2$

$\sigma(u,v) = (u, v, u^2 + v^2)$.

Should get $K > 0$ everywhere.

$\boxed{6A}$

Elliptic paraboloid

$$\sigma(u,v) = (u, v, u^2+v^2)$$

$$F : x^2+y^2-z$$

$$\nabla F = (2u, 2v, -1). \quad |\nabla F| = \frac{1}{\sqrt{1+4u^2+4v^2}}$$

$$\sigma_u = (1, 0, 2u)$$

$$\sigma_v = (0, 1, 2v)$$

$$\sigma_{uu} = (0, 0, 2)$$

$$\sigma_{uv} = 0$$

$$\sigma_{vv} = (0, 0, 2)$$

$$E = G = 5. \quad F = 0$$

$$e = \frac{1}{|\nabla F|} \nabla F \cdot \sigma_{uu} = \frac{-2}{\sqrt{1+4u^2+4v^2}}$$

$$f = \frac{1}{|\nabla F|} \nabla F \cdot \sigma_{uv} = 0$$

$$g = \frac{1}{|\nabla F|} \nabla F \cdot \sigma_{vv} = \frac{-2}{\sqrt{1+4u^2+4v^2}}$$

$$\text{So } K = \frac{eg-f^2}{EG-F^2} = \frac{4}{(1+4u^2+4v^2)}$$

OMIT 2-8

9. New surface $\varphi = c \cdot \sigma$.

Compute FFF $\varphi = c^2 \cdot$ FFF σ

What about SFF?

Tangent plane to φ at parameter (u,v) goes through $c \cdot \sigma(u,v)$, normal direction $=$ $c\sigma_1 \times c\sigma_2$ is same direction as for σ, so $N_\varphi = N_\sigma$. Hence SFF $(\varphi) = c \cdot$ SFF(σ).

So
$$H_\varphi = \frac{e_\varphi G_\varphi - 2 f_\varphi F_\varphi + g_\varphi E_\varphi}{2(E_\varphi G_\varphi - F_\varphi^2)} =$$

$$\frac{c^3(e_\sigma G_\sigma - 2 f_\sigma F_\sigma + g_\sigma E_\sigma)}{c^4 \cdot 2 \cdot (E_\sigma G_\sigma - F_\sigma^2)} =$$

$$\frac{1}{c} H_\sigma.$$

Likewise,

$$K_\varphi = \frac{e_\varphi g_\varphi - f_\varphi^2}{E_\varphi G_\varphi - F_\varphi^2} = \frac{c^2(e_\sigma g_\sigma - f_\sigma^2)}{c^4(E_\sigma G_\sigma - F_\sigma)^2} =$$

$$\frac{1}{c^2} K_\sigma.$$

12. (a) If $\det A > 0$ then either $\left\{ \begin{matrix} \lambda_1, \lambda_2 < 0 \\ \text{or } \lambda_1, \lambda_2 > 0 \end{matrix} \right\}$

Hence the $\left\{ \begin{matrix} \text{maximum} \\ \text{minimum} \end{matrix} \right\}$ of the Rayleigh quotient is $\left\{ \begin{matrix} \text{negative} \\ \text{positive} \end{matrix} \right\}$ and the quotient is never zero.

(b) $\det A < 0 \Rightarrow$ have eigenvalues with $\lambda_2 < 0 < \lambda_1$. Let $u_1 + u_2$ be the corresponding eigenvectors. Then X is asymptotic $\Longleftrightarrow X = t_1 u_1 + t_2 u_2$ with

$$\lambda_1 t_1^2 + \lambda_2 t_2^2 = 0 \text{ or } \lambda_1 t_1^2 = |\lambda_2| t_2^2.$$

this holds $\Longleftrightarrow$ $t_1 = \pm\sqrt{\frac{|\lambda_2|}{\lambda_1}}\, t_2.$

We get lin indep vectors by taking $t_2 = 1$.

(c) Find the cosine of the angle between

$$\underbrace{\sqrt{\frac{|\lambda_2|}{\lambda_1}}\, w_1 + w_2}_{p} \quad \text{and} \quad \underbrace{-\sqrt{\frac{|\lambda_2|}{\lambda_1}}\, w_1 + w_2}_{q}:$$

$$\frac{p \cdot q}{|p||q|} = \frac{1 - \frac{|\lambda_2|}{\lambda_1}}{1 + \frac{|\lambda_2|}{\lambda_1}} = \frac{\lambda_1 - |\lambda_2|}{\lambda_1 + |\lambda_2|}.$$

$$\Rightarrow |\text{cos of angle}| = \frac{|\lambda_1 - |\lambda_2||}{\lambda_1 + |\lambda_2|}.$$

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